Path integral measure as determined by canonical gravity

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Motivation

- **Dynamics** of current spin foam approach is independent from canonical theory
- Need to use **Liouville measure** on the reduced phase space which preserves the phase space volume

$$\int \mathcal{D}q \exp iS \to \int \mathcal{D}q\mathcal{D}p \exp iS$$

- **Plebanski-Holst formulation:**

  has desired variables $(\omega^{IJ}_{\mu\nu}, X^{IJ}_{\mu\nu}) \longrightarrow$ need to integrate out the connection

Outline

- Reduced phase space path integral approach for a general Hamiltonian system with constraints
- Applying that to Plebanski Holst formulation
- Integrating out the connection
- ADM path integral
- Conclusion

Reduced phase space path integral

(Quantization of Gauge Systems, Henneaux and Teitelboim)

- First class constraints: $\{F_i, F_j\} = f_{ij}^k F_k \approx 0$
- Second class constraints: $\{S_i, S_j\} \not\approx 0$

Liouville measure on the whole phase space $dqdp$

Liouville measure on the reduced phase space $dq^\star dp^\star$

Liouville measure for noncanonical coordinates in phase space: $[\det\{x^\mu, x^\nu\}]^{-\frac{1}{2}} dx^\mu$

- Change of variables: $(q, p) \to (q^\star, p^\star, S)$

$$dq^\star dp^\star dS [\det\{S, S\}]^{-\frac{1}{2}} = dqdp \quad \longrightarrow \quad \int dq^\star dp^\star = \int \delta(S) [\det\{S, S\}]^{\frac{1}{2}} dqdp$$

Reduced phase space path integral

(Quantization of Gauge Systems, Henneaux and Teitelboim)

Gauge fixing: $(F, \xi) \rightarrow$ second class

$$\int dq^\star dp^\star = \int \delta(S)[\det\{S,S\}]^{\frac{1}{2}} dq dp$$

$$\begin{Bmatrix} \{F,F\} & \{F,\xi\} \\ \{\xi,F\} & \{\xi,\xi\} \end{Bmatrix} \longrightarrow [\det\{F,\xi\}]^2$$

Faddeev Popov term:

$$[\det\{S,S\}]^{\frac{1}{2}} \rightarrow \Delta_{FP} = |\det(\{F,\xi\})|$$

$$\delta(S) \rightarrow \delta(F)\delta(\xi)$$

Path integral:

$$\mathcal{Z} := \int \mathcal{D}q\mathcal{D}p \exp iS$$

$$\mathcal{Z} := \int \mathcal{D}q\mathcal{D}p \sqrt{\det(\{S,S\})} |\det(\{F,\xi\})| \delta[S]\delta[F]\delta[\xi] \exp iS$$

Expectation value:

$$\mathcal{Z}(\mathcal{O}) := \int \mathcal{D}q\mathcal{D}p \sqrt{\det(\{S,S\})} |\det(\{F,\xi\})| \mathcal{O}\delta[S]\delta[F]\delta[\xi] \exp iS$$

Plebanski Holst formulation

- P-H action:

$$S_{PH} = \int \overset{(\gamma)}{X}_{IJ} \wedge F^{IJ}$$

$$\overset{(\gamma)}{X}{}^{IJ} = (X - \frac{1}{\gamma} \star X)^{IJ}$$

$$F^{IJ} = d\omega^{IJ} + \omega^{I}{}_{K} \wedge \omega^{KJ}$$

- Conjugate variables: $(\omega^{IJ}_{\mu\nu}, X^{IJ}_{\mu\nu})$

- Simplicity constraint:

$$C_{\mu\nu\rho\sigma} := \epsilon_{IJKL} X^{IJ}_{\mu\nu} X^{KL}_{\rho\sigma} - \frac{s}{4!} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} \epsilon_{IJKL} X^{IJ}_{\alpha\beta} X^{KL}_{\gamma\delta} \approx 0$$

$$X^{IJ} = \begin{cases} \pm \frac{1}{\kappa} e^I \wedge e^J & (I\pm) \\ \pm \frac{1}{2\kappa} \epsilon^{IJ}{}_{KL} e^K \wedge e^L & (II\pm) \end{cases}$$

- Reduced phase space integral[1]:

$$\mathcal{Z} = \int_{(II+)} \mathcal{D}\omega^{IJ}_{\mu} \mathcal{D}X^{IJ}_{\mu\nu} \delta^{20}(C) \mathcal{V}^9 V_s \exp i \int \overset{(\gamma)}{X}_{IJ} \wedge F^{IJ}$$

$$\mathcal{V}_s = h^{\frac{1}{2}}$$

$$\mathcal{V} = g^{\frac{1}{2}}$$

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1) Canonical path integral measures for Holst and Plebanski gravity: I. Reduced phase space derivation, J. Engle, M. Han and T. Thiemann, 2010 Class. Quantum Grav. 27 245014

Integrating out the connection

$$S[X,\omega] = (\omega, \hat{A}\omega) + (b,\omega)$$

$$(\alpha,\beta) := \int d^4x \delta^{\mu\nu} \alpha^{IJ}_\mu \beta_{\nu IJ}$$

Gaussian integral:

$$I(X) = \int \mathcal{D}\omega \exp i \left[(\omega, \hat{A}\omega) + (b,\omega) \right]$$

$$\hookrightarrow \quad I(X) \hat{=} (\det \hat{A})^{-\frac{1}{2}} \exp \frac{-i}{4} (b, \hat{A}^{-1} b)$$

Determinant of A: $\det \hat{A} = \mathcal{V}^{12}$

$$\mathcal{Z} = \int_{(II+)} \mathcal{D}\omega^{IJ}_\mu \mathcal{D}X^{IJ}_{\mu\nu} \delta^{20}(C) \mathcal{V}^9 V_s \exp iS$$

$$\hookrightarrow \quad \mathcal{Z}^{(1)} \hat{=} \int \mathcal{D}X^{IJ}_{\mu\nu} \delta^{20}(C) \mathcal{V}^3 V_s \exp iS$$

1) Pure momentum path integral for spin foam, A. Ch. Shirazi & J. Engle (In Preparation)

ADM Formulation

- Canonical variables: (h_{ab}, π^{ab}) $\qquad \pi^{ab} = \dfrac{\partial \mathcal{L}}{\partial \dot{h}_{ab}}$

- First class constraints: Hamiltonian H_0 and 3-diff constraints H_a

$$H_0 = -\frac{h^{-\frac{1}{2}}}{2}[h_{ac}h_{bd} + h_{ad}h_{bc} - h_{ab}h_{cd}]\pi^{ab}\pi^{cd} - {}^{(3)}R(h)h^{1/2}$$
$$H_a = h^{-\frac{1}{2}}h_{ab}D_c\pi^{bc}$$

- No second class constraints

$$\mathcal{Z} := \int \mathcal{D}q\mathcal{D}p\sqrt{\det(\{S,S\})}\Delta_{FP}\delta[S]\delta[F]\delta[\xi]\exp iS$$

$$\hookrightarrow \quad \mathcal{Z}_{ADM} = \int \mathcal{D}h_{ab}\mathcal{D}\pi^{ab}\mathcal{D}N\mathcal{D}N^a \exp i\int d^4x(\pi^{ab}\dot{h}_{ab} - \mathcal{H}_G(h_{ab},\pi^{ab}))$$

Using: $\quad \{\int \mathcal{D}N \exp -iNH = \delta(H), \qquad \int \mathcal{D}N^a \exp -iN^aH_a = \delta(H_a)\}$

- N and N^a are Lagrange multipliers
- No gauge fixing: No Faddeev-popov term

Final ADM path integral

$$\mathcal{Z}_{ADM} = \int \mathcal{D}h_{ab}\mathcal{D}\pi^{ab}\mathcal{D}N\mathcal{D}N^a \exp i \int dt(\hat{A}_{ab,cd}\pi^{ab}\pi^{cd} + \hat{B}_{ab}\pi^{ab} + \hat{C})$$

$$\hat{A}_{ab,cd} := -\frac{Nh^{-\frac{1}{2}}}{2}[h_{ac}h_{bd} + h_{ad}h_{bc} - h_{ab}h_{cd}]$$
$$\hat{B}_{ab} := \dot{h}_{ab} - 2D_{(a}N_{b)}$$
$$\hat{C} := Nh^{\frac{1}{2}}\,{}^{(3)}\mathrm{R}$$

- Integrating out π^{ab}

$$\mathcal{Z}_{ADM} = \int (\det \hat{A})^{-\frac{1}{2}}\mathcal{D}h_{ab}\mathcal{D}N\mathcal{D}N^a \exp i \int d^4x(\frac{-1}{4}\hat{B}_{ab}\hat{A}^{-1ab,cd}\hat{B}_{cd} + \hat{C})$$

The exponential: $(\frac{-1}{4}\hat{B}_{ab}\hat{A}^{-1ab,cd}\hat{B}_{cd} + \hat{C}) \to \mathcal{L}_G$ Determinant: $(\det \hat{A})^{-\frac{1}{2}} = N^{-3}h^{-\frac{1}{2}}$

- Final ADM path integral[1]

$$\mathcal{Z}_{ADM} \hat{=} \int N^{-3}h^{-\frac{1}{2}}\mathcal{D}h_{ab}\mathcal{D}N\mathcal{D}N^a \exp i \int d^4x\mathcal{L}_G.$$

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Comparing to our results

$$(1)\quad \mathcal{Z}_{ADM} \hat{=} \int N^{-3} h^{-\frac{1}{2}} \mathcal{D}h_{ab} \mathcal{D}N \mathcal{D}N^a \exp iS_G$$

$$(2)\qquad \mathcal{Z} \hat{=} \int \mathcal{D}X^{IJ}_{\mu\nu} \delta^{20}(C) \mathcal{V}^3 V_s \exp iS_G$$

- Change of variables $(h_{ab}, N, N^a) \to (g_{\mu\nu})^{(1)}$

$$\mathcal{D}g_{\mu\nu} = \mathcal{D}g_{ab}\mathcal{D}g_{00}\mathcal{D}g_{0a} = \det J \mathcal{D}h_{ab}\mathcal{D}N\mathcal{D}N^a$$

$$g_{ab} = h_{ab}$$
$$g_{00} = -N^2 + h_{ab}N^aN^b$$
$$g_{0a} = h_{ab}N^b$$

$$J = \frac{\partial(g_{ab}, g_{00}, g_{0a})}{\partial(h_{ab}, N, N^a)} =$$

$\frac{\partial(g_{ab},g_{00},g_{0a})}{\partial(h_{ab},N,N^a)}$	g_{ab}	g_{00}	g_{0a}
N	0	$-2N$	0
N^a	0	$2N^a$	h_{ab}
h_{ab}	1	0	N^a

$$\det J = -2hN$$

$$\mathcal{D}g_{\mu\nu} = -2hN\mathcal{D}h_{ab}\mathcal{D}N\mathcal{D}N^a \longrightarrow \boxed{\mathcal{Z}_{ADM} \hat{=} \int N^{-4} h^{-\frac{3}{2}} \mathcal{D}g_{\mu\nu} \exp iS_G}$$

- Other change of variables:

$$X^{IJ}_{\mu\nu} \to (e^I_\mu, C)^{(2)} \qquad \mathcal{D}X^{IJ}_{\mu\nu} = \mathcal{V}^{-6}\mathcal{D}e^I_\mu \mathcal{D}^{20}C$$

$$(e^I_\mu) \to (g_{\mu\nu}, \Lambda_I{}^J)^{(3)} \qquad \mathcal{D}g_{\mu\nu}\mathcal{D}\Lambda_I{}^J = \sqrt{g}\mathcal{D}e^I_\mu$$

1) Pure momentum path integral for spin foam, A. Ch. Shirazi & J. Engle (In Preparation)
2) Canonical path integral measures for Holst and Plebanski gravity: I. Reduced phase space derivation, J. Engle, M. Han and T. Thiemann, 2010 Class. Quantum Grav. 27 245014
3) Path integral measure for first-order and metric gravities, R. Aros, M. Contreras and J. Zanelli, 2003 *Class. Quantum Grav.* **20** 2937

Conclusion

- Breaking the manifest general covariance because of the **appearance of 3-volume**.

  Reason: Foliation of space-time! Price to pay for consistency with the canonical theory.

- The path integral is invariant under **Bergman-Komar "group"** but not Diff(M)!
  (Canonical Path-Integral Measures for Holst and Plebanski Gravity. II. Gauge Invariance and Physical Inner Product, M. Han, Class.Quant.Grav. 27 (2010) 245015)

- **Continuum path integral**

  Next step: Discretizing and quantizing the measure and implying to spin foam models.