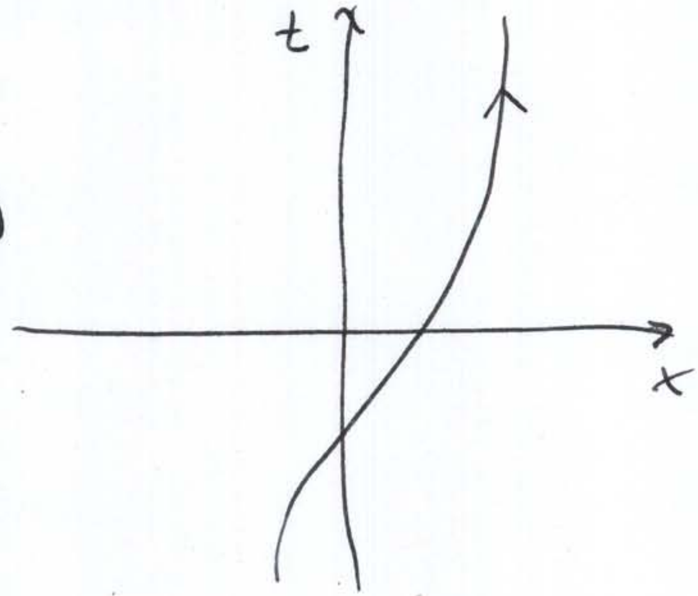
Relativistic Dynamics

Doubly
Special
Relativity (DSR)



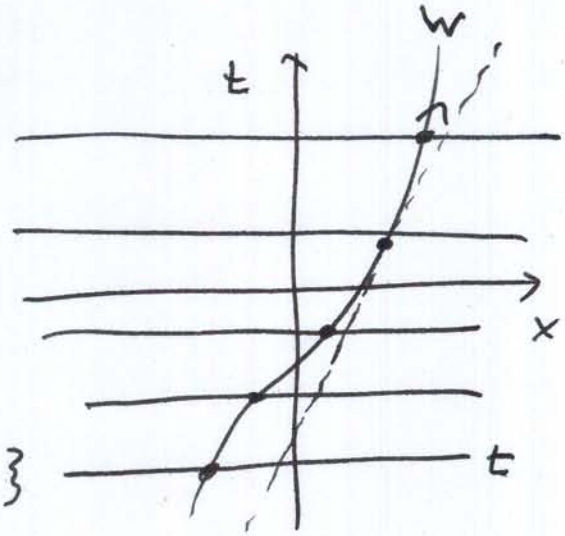
Inputs:

- low velocities
  $v \ll c$

  $\leadsto$ Newton's laws

- assume: everything transforms
  covariantly under <u>Lorentz</u>
  ↑
  (DSR)

Four - Velocity



world-line

= {events the particle goes through}

↑

no natural parameterization

(Newtonian: Universal time t)

In any inertial coordinates:

$$\underset{\substack{\uparrow \\ \text{3-vector}}}{\vec{x}(t)} \rightsquigarrow \underset{\substack{\uparrow \\ \text{4-vector (param. by } t)}}{w(t)} \underset{O}{\rightarrow} \begin{pmatrix} t \\ \vec{x}(t) \end{pmatrix} = \begin{pmatrix} t \\ x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

Reparameterization:

$$\underbrace{\underset{\substack{\uparrow \\ O}}{w(t)} \equiv \underset{\substack{\uparrow \\ O'}}{w(t')}}_{\text{physically equivalent}} \equiv \underset{\substack{\uparrow \\ \lambda \to \lambda' = \lambda'(\lambda)}}{w(\lambda)}$$

Velocity $\rightsquigarrow$ derivative w.r.t. "time"

$\uparrow$ parameter λ

$$u = \frac{dw(\lambda)}{d\lambda}$$

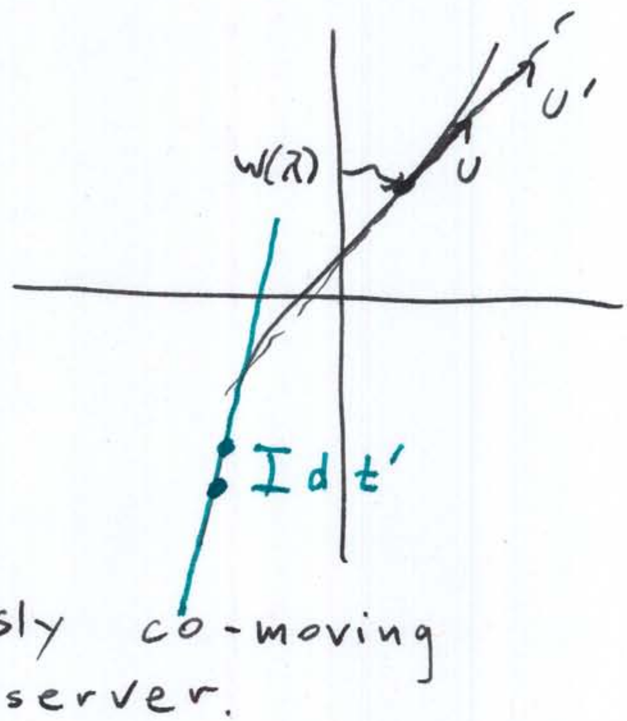
$$\lambda \to \lambda' = \lambda'(\lambda)$$

$$u \to u' = \frac{dw}{d\lambda'} = \frac{d\lambda}{d\lambda'}\frac{dw}{d\lambda} = \left(\frac{d\lambda'}{d\lambda}\right)^{-1}\frac{dw}{d\lambda}$$

$\uparrow$ parallel, but not $=$, to u

$\uparrow$ scaling factor

Proper Time



definition:

$d\tau(\lambda)$

$\|$

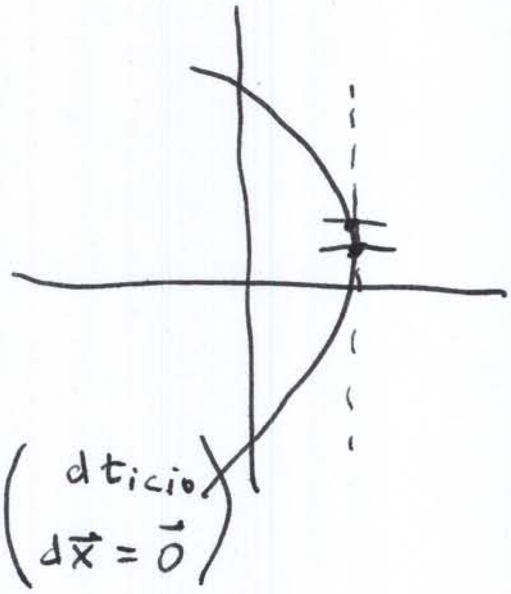
$d\,t_{icio}(\lambda)$

$\uparrow$

instantaneously co-moving inertial observer.

Proper time is the natural parameterization for time-like curves.

$$u = \frac{dw}{d\tau}$$



$$d\tau := dt_{icio}$$

$\cancel{w(\tau) \quad w(\tau + d\tau)}$

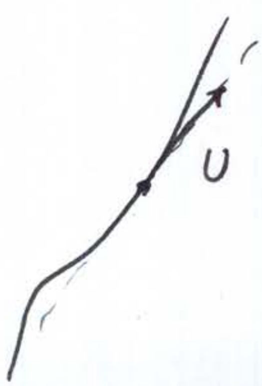
$$w(\tau + d\tau) - w(\tau) \underset{icio}{=} \begin{pmatrix} dt_{icio} \\ d\vec{x} = \vec{0} \end{pmatrix}$$

$$\rightsquigarrow \|dw\|^2 = -c^2 dt_{icio}^2 = -c^2 d\tau^2$$

$$\rightsquigarrow \left\| \frac{dw}{d\tau} \right\|^2 = -c^2$$

$\uparrow$

four-velocity (in τ-param.) ~~is~~ has fixed norm $-c^2$

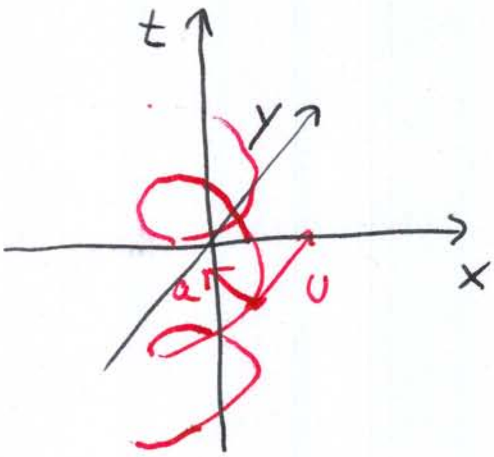


$$\|u\|^2 = -c^2$$

Four - Acceleration

$$a = \frac{dU(\tau)}{d\tau}$$

Example: particle moving in circle of radius r in the inertial coordinates of O: (const. Ω)



$$W(t) = \begin{pmatrix} t \\ r\cos(\Omega t) \\ r\sin(\Omega t) \end{pmatrix}$$

$$\frac{dW}{dt} = \begin{pmatrix} 1 \\ -\Omega r \sin(\Omega t) \\ \Omega r \cos(\Omega t) \end{pmatrix}$$

tangent to world-line

$$\left\| \frac{dW}{dt} \right\|^2 = -c^2 + (\underbrace{\Omega r}_{v})^2 \sin^2(\Omega t) + (\Omega r)^2 \cos^2(\Omega t)$$

$$= v^2 - c^2$$

$$\frac{c^2}{c^2 - v^2} \left\| \frac{dW}{dt} \right\|^2 = -c^2 \quad \rightsquigarrow \quad U = \sqrt{\frac{1}{1 - v^2/c^2}} \, \frac{dW}{dt}$$

$$u = \gamma \begin{pmatrix} 1 \\ -\Omega r \sin(\Omega t) \\ \Omega r \cos(\Omega t) \end{pmatrix} \qquad u^0 = \frac{dt}{d\tau} = \gamma$$

$$a = \frac{du}{d\tau} = \left(\frac{d\tau}{dt}\right)^{-1} \frac{du}{dt} = \gamma \frac{du}{dt}$$

$$= \gamma \cdot \gamma \cdot \begin{pmatrix} 0 \\ -\Omega^2 r \cos(\Omega t) \\ -\Omega^2 r \sin(\Omega t) \end{pmatrix}$$

$$\frac{d}{d\tau}\left(\|u\|^2 = -c^2 \right)$$

$$= \frac{d}{d\tau} u \cdot u = 2u \cdot \frac{du}{d\tau} = 2u \cdot a = 0$$

$\Rightarrow$ $a \perp u$ at all times τ

four-acceleration is <u>always</u> orthogonal to four-velocity

Scalar Force

1) momentum: $p = mU$

2) Newton's 2^{nd} Law

$$\Rightarrow \frac{dp}{d\tau} = \sum F$$

four-force.

Non-relativistic physics

$$\vec{f} = -q \vec{\nabla} \psi$$

"scalar charge"

$$F_\alpha = -q \nabla_\alpha \psi = -q \frac{\partial \psi}{\partial x^\alpha}$$

$$\dot{p}_\alpha = -q \nabla_\alpha \psi$$

$$\Rightarrow U^\alpha \dot{p}_\alpha = -q U^\alpha \nabla_\alpha \psi \quad \neq 0 \ !!!$$

$$U^\alpha \frac{d}{d\tau} p_\alpha = \frac{d}{d\tau}(U^\alpha p_\alpha) - p_\alpha \frac{dU^\alpha}{d\tau}$$

mU_α

$$= \frac{d}{d\tau}(-mc^2) - 0$$

$$U^\alpha \nabla_\alpha \psi = \frac{dw^\alpha}{d\tau} \nabla_\alpha \psi = \frac{d\psi}{d\tau}$$

$$-c^2 \frac{dm}{d\tau} = -q \frac{d\psi}{d\tau}$$

m <u>not</u> constant!

$$\rightsquigarrow \quad m = \boxed{m_0 + \frac{q}{c^2}\psi}$$

$$\dot{p}_\alpha = -q \nabla_\alpha \psi$$

Electromagnetic Forces

$$\vec{f} = q(\vec{E} + \vec{v} \times \vec{B}/c)$$

tensor field $F_{\alpha\beta}$

$$F_{\alpha\beta} = \left(\begin{array}{c|c} 0 & -\vec{E} \\ \hline +\vec{E} & -\vec{B}\times \end{array}\right)$$

3×3 matrix

$$(\vec{B}\times)\cdot\vec{v} = \vec{B}\times\vec{v}$$

$$\boxed{F_\alpha = qF_{\alpha\beta}U^\beta} = \begin{pmatrix} 0 & -\vec{E} \\ \vec{E} & -\vec{B}\times \end{pmatrix}\begin{pmatrix} \gamma \\ \gamma\vec{v} \end{pmatrix} = \begin{pmatrix} -\gamma\vec{E}\cdot\vec{v} \\ \gamma\vec{E} - \gamma\vec{B}\times\vec{v} \end{pmatrix}$$

$$F_\alpha = q\, F_{\alpha\beta}\, U^\beta$$

symmetric

$$U^\alpha F_\alpha = q\, F_{\alpha\beta} \left(U^\alpha U^\beta \right) = 0$$

anti-symmetric