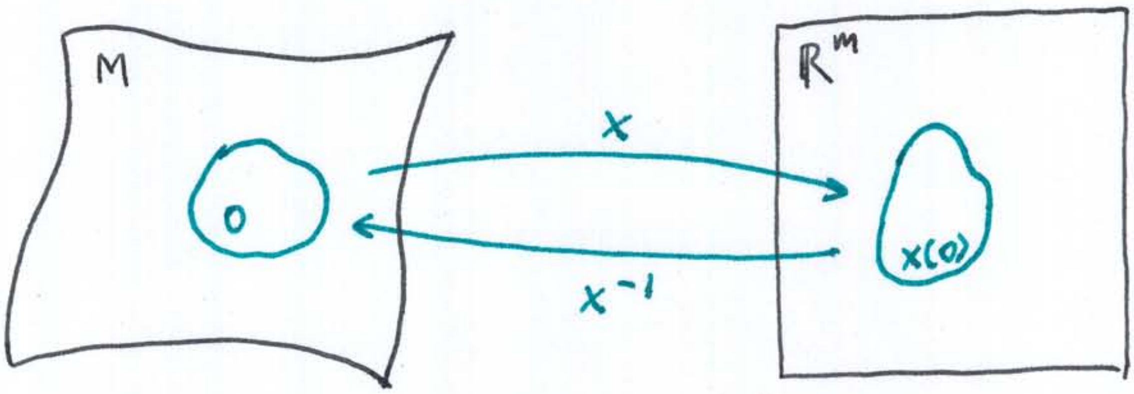
Lecture 7

Differential Geometry

How do I do my homework?

Coordinate Charts

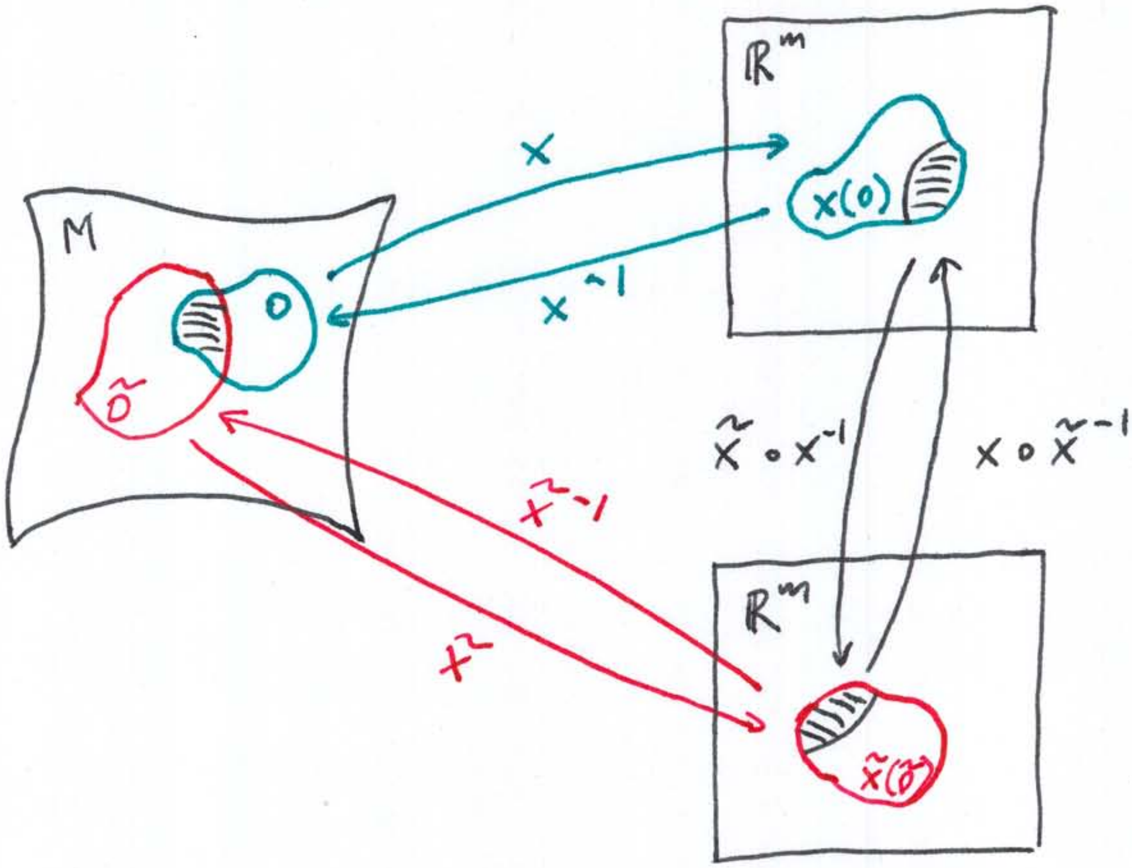


A coordinate chart on a manifold M is a pair (x, O), where

- O is an (open) subset of M
- x is a 1-1 mapping from O to an (open) subset of $\mathbb{R}^m$

This mapping is invertible, but often does not cover all of M.

Compatible Charts



Two charts (x, O) and $(\tilde{x}, \tilde{O})$ are said to be <u>compatible</u> if, on the set $O \cap \tilde{O} \subset M$ where both are defined, the maps

$$\tilde{x} \circ x^{-1} \text{ and } x \circ \tilde{x}^{-1}$$

from $\mathbb{R}^m$ to $\mathbb{R}^m$ are <u>smooth</u>.

Manifolds

A set M is called a <u>smooth manifold</u> if it can be equipped with an <u>atlas</u>

$$\mathcal{A} = \{ (x_i, O_i) \}$$

of coordinate charts such that:

- Every chart is compatible with all of the others.
- Every point of M is covered by at least one chart.
- Every chart (x, O) on M that is compatible with all of the charts in $\mathcal{A}$ is itself in $\mathcal{A}$.

(maximal atlas ⇝ Universality)

Example: $\mathbb{R}^m$ (trivial)

One global chart:

$$O = \mathbb{R}^m \; ; \quad x : \mathbb{R}^m \to \mathbb{R}^m = \text{identity}$$

$\rightsquigarrow \mathcal{A}$ contains <u>all</u> smooth coordinate systems on $\mathbb{R}^m$ (spherical, cylindrical, etc.)

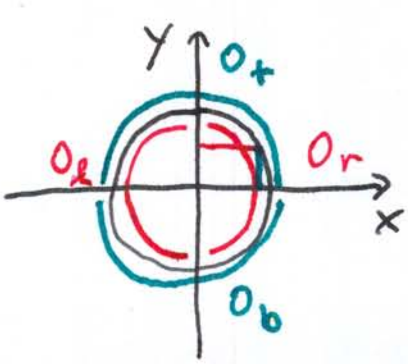
Example: S^1 (circle)

$$S^1 = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$$

Four charts:

$O_+ = \{ y > 0 \}$ $\qquad$ $O_l = \{ x < 0 \}$

$O_b = \{ y < 0 \}$ $\qquad$ $O_r = \{ x > 0 \}$



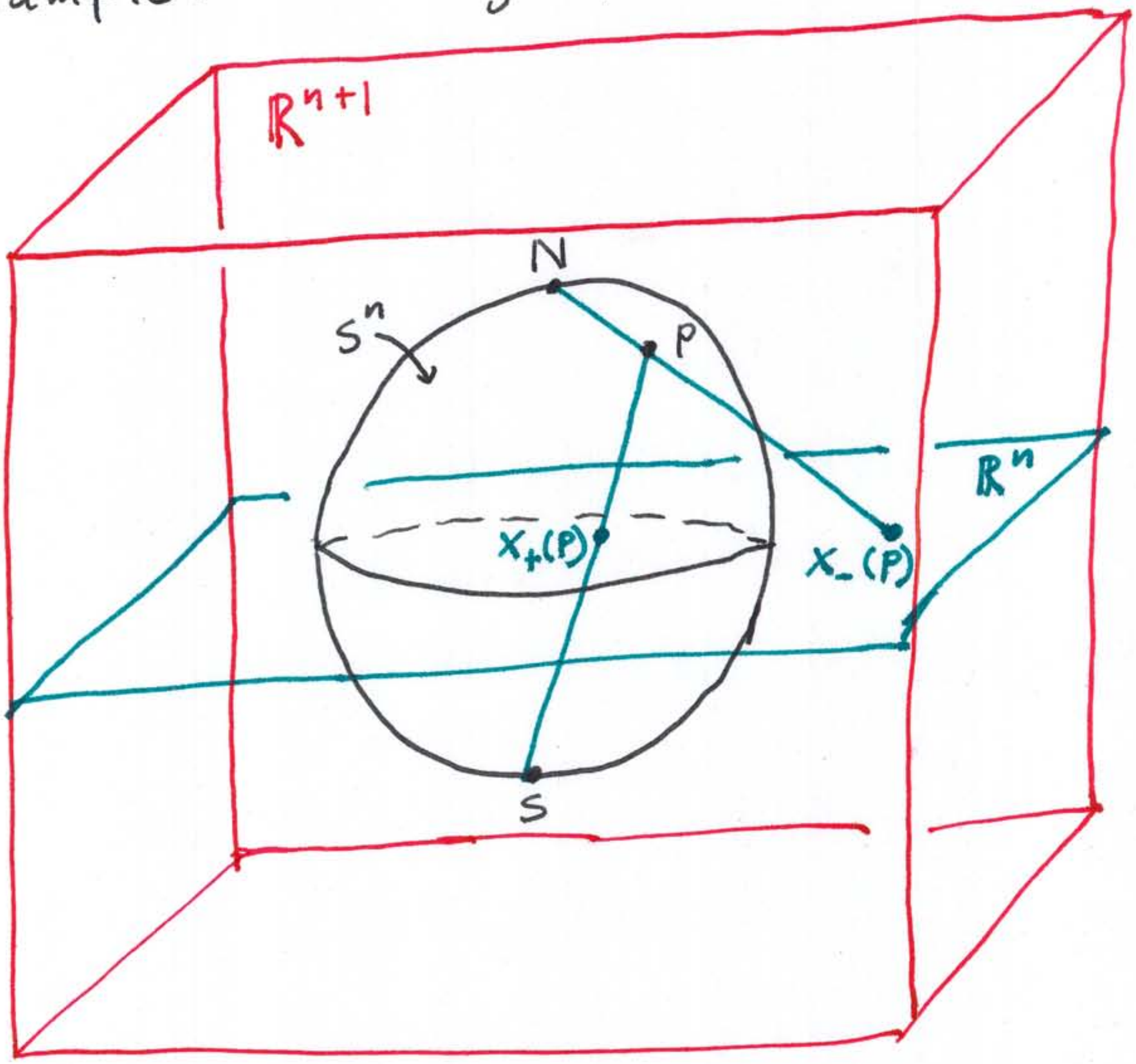
$$z_+ = x = z_b$$

$$z_l = y = z_r$$

$$z_+ \circ z_r^{-1}(y) =$$

$$= z_+(\sqrt{1-y^2}, y) = \sqrt{1-y^2}$$

Example: Stereographic Projection



$$S^n = \{(x^0, \dots, x^n) \mid (x^0)^2 + \dots + (x^n)^2 = 1\}$$

North pole N : $x^0 = +1$, $\vec{x} = 0$

South pole S : $x^0 = -1$, $\vec{x} = 0$

$(x^0, \vec{x})$

$$X_\pm(x^0, \vec{x}) := \frac{\vec{x}}{1 \pm x^0} = \vec{\xi}_\pm$$

n-dim. coordinate on S^n

Need to show $X_\pm \circ X_\mp$ is smooth in the ordinary sense.

$$X_\pm^{-1}(\vec{\xi}): \quad \vec{\xi} = \frac{\vec{x}}{1 \pm x^0}$$

$$\Rightarrow \|\vec{\xi}\|^2 = \frac{\|\vec{x}\|^2}{(1 \pm x^0)^2} = \frac{1-(x^0)^2}{(1 \pm x^0)^2}$$

$$= \frac{1 \mp x^0}{1 \pm x^0}$$

$$\Rightarrow x^0 = \pm \frac{1 - \|\vec{\xi}\|^2}{1 + \|\vec{\xi}\|^2}$$

$$\Rightarrow \vec{x} = (1 \pm x^0)\, \vec{\xi} = \frac{2\vec{\xi}}{1 + \|\vec{\xi}\|^2}$$

$$X_\pm^{-1}(\vec{\xi}) = \left(\pm \frac{1 - \|\vec{\xi}\|^2}{1 + \|\vec{\xi}\|^2}, \frac{2\vec{\xi}}{1 + \|\vec{\xi}\|^2} \right)$$

$$\Rightarrow X_\pm \circ X_\mp^{-1}(\vec{\xi}) = X_\pm \left(\mp \frac{1 - \|\vec{\xi}\|^2}{1 + \|\vec{\xi}\|^2}, \frac{2\vec{\xi}}{1 + \|\vec{\xi}\|^2} \right)$$

$$= \left(1 \pm \left(\mp \frac{1 - \|\vec{\xi}\|^2}{1 + \|\vec{\xi}\|^2} \right) \right)^{-1} \frac{2\vec{\xi}}{1 + \|\vec{\xi}\|^2}$$

$$= \left(1 - \frac{1 - \|\vec{\xi}\|^2}{1 + \|\vec{\xi}\|^2} \right)^{-1} \frac{2\vec{\xi}}{1 + \|\vec{\xi}\|^2}$$

$$\frac{2\|\vec{\xi}\|^2}{1 + \|\vec{\xi}\|^2}$$

$$= \frac{\vec{\xi}}{\|\vec{\xi}\|^2}$$

So, $x_{\pm} \circ x_{\mp}^{-1} : \vec{\xi} \mapsto \dfrac{\vec{\xi}}{\|\vec{\xi}\|^2}$ is smooth except at the origin.

But, x_+ is undefined at S.

x_- is undefined at N.

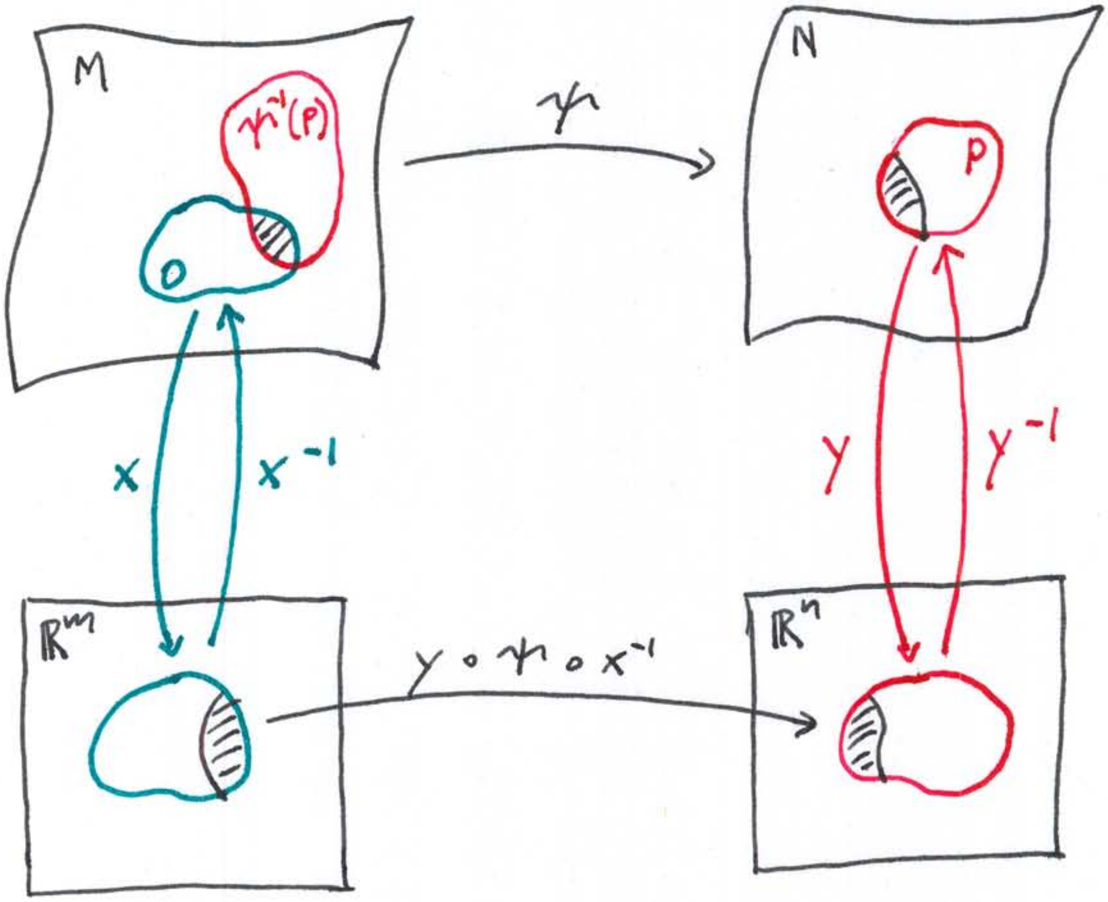
The <u>overlap functions</u> need to be smooth only where both charts are defined

- $x_+(N) = 0 = x_-(S)$

Therefore, the singularity at the origin is ok.

S^n is a manifold (covered by two charts)

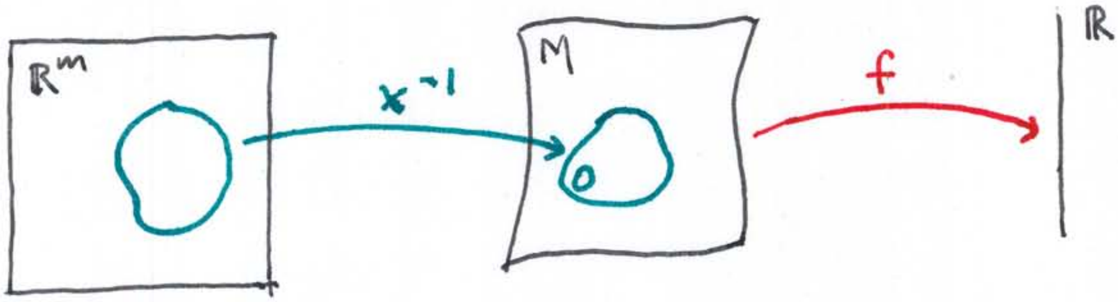
Smooth Mappings



A mapping $\psi : M \to N$ between smooth manifolds is <u>smooth</u> if, for any charts (x, O) on M and (y, P) on N, the composed map $y \circ \psi \circ x^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth (wherever it is defined.)

Example: Smooth Function

A <u>smooth function</u> is a smooth mapping from a manifold M to the manifold $\mathbb{R}$.



$$f \circ x^{-1} : (x^1, \dots, x^m) \mapsto f(p)$$

$$\rightsquigarrow f_x(x^1, \dots, x^m) = (\text{number})$$

$\uparrow$

This function on $\mathbb{R}^m$ must be <u>smooth</u> (infinitely continuously differentiable) in the ordinary sense of calculus in m variables.

$$\frac{\partial f_x}{\partial x^1}, \dots, \frac{\partial f_x}{\partial x^m}$$

$$\frac{\partial^2 f_x}{\partial x^1 \partial x^2}, \text{ etc.}, \text{ etc.}$$

Note: Consistent coordinate charts lead to consistent definitions of smooth functions.

$$(x, \mathcal{O}) \rightsquigarrow f_x = f \circ x^{-1}$$

$$(\tilde{x}, \tilde{\mathcal{O}}) \rightsquigarrow f_{\tilde{x}} = f \circ \tilde{x}^{-1}$$

$$= f \circ x^{-1} \circ x \circ \tilde{x}^{-1}$$

$$= f_x \circ (x \circ \tilde{x}^{-1})$$

In ordinary notation,

$$f_{\tilde{x}}(\tilde{x}^1, \ldots, \tilde{x}^m) =$$

$$f_x(x^1(\tilde{x}^1, \ldots, \tilde{x}^m), \ldots, x^m(\tilde{x}^1, \ldots, \tilde{x}^m))$$

$f_x : \mathbb{R}^m \to \mathbb{R}$, smooth

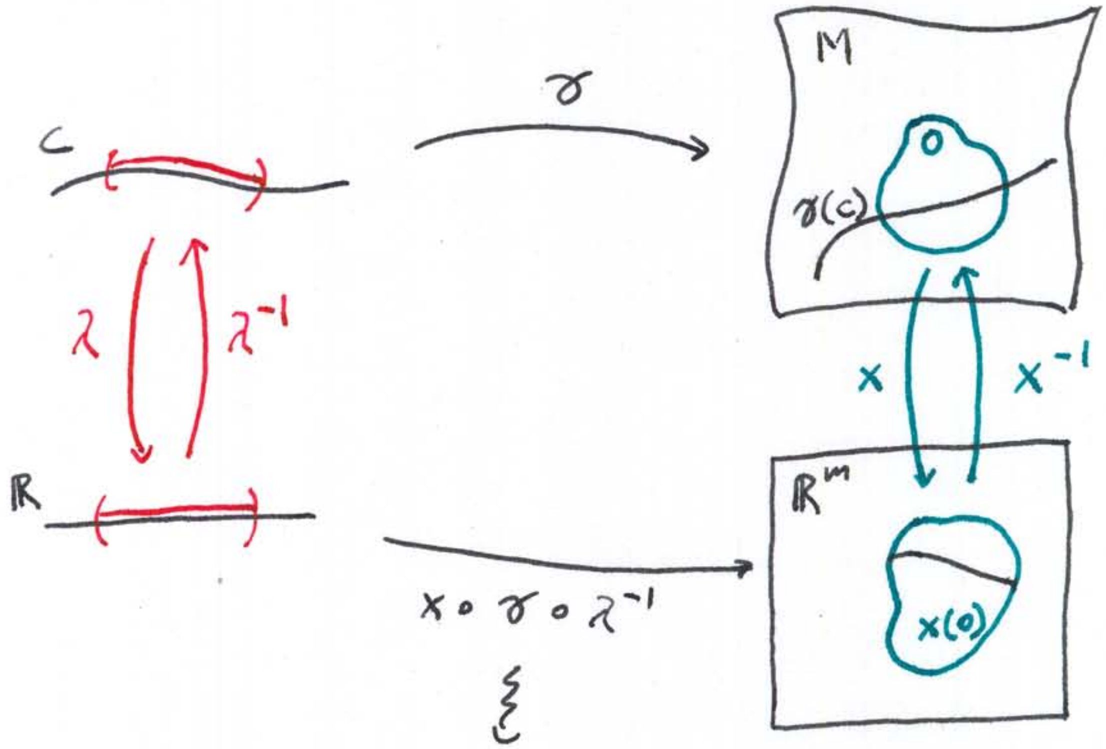
$x \circ \tilde{x}^{-1} : \mathbb{R}^m \to \mathbb{R}^m$, smooth

$\Rightarrow f_{\tilde{x}} : \mathbb{R}^m \to \mathbb{R}$ is smooth

by the chain rule!

Example: Smooth Curve

A <u>smooth curve</u> is a smooth mapping γ from a one-dim. manifold C into a m-dim. manifold M.



$(x^1(\lambda), \ldots, x^m(\lambda))$

$\uparrow$

smooth curve in $\mathbb{R}^m$

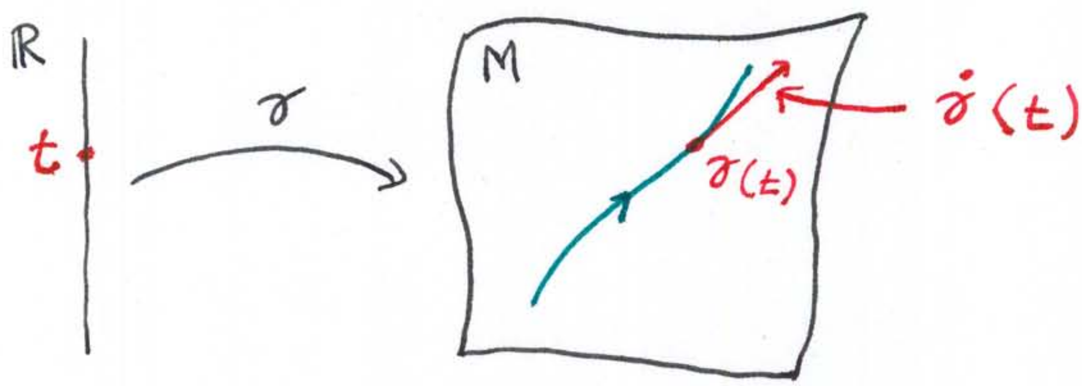
Note: this definition does not assume a <u>parameterization</u>.

$(\mathbb{R} \to M)$

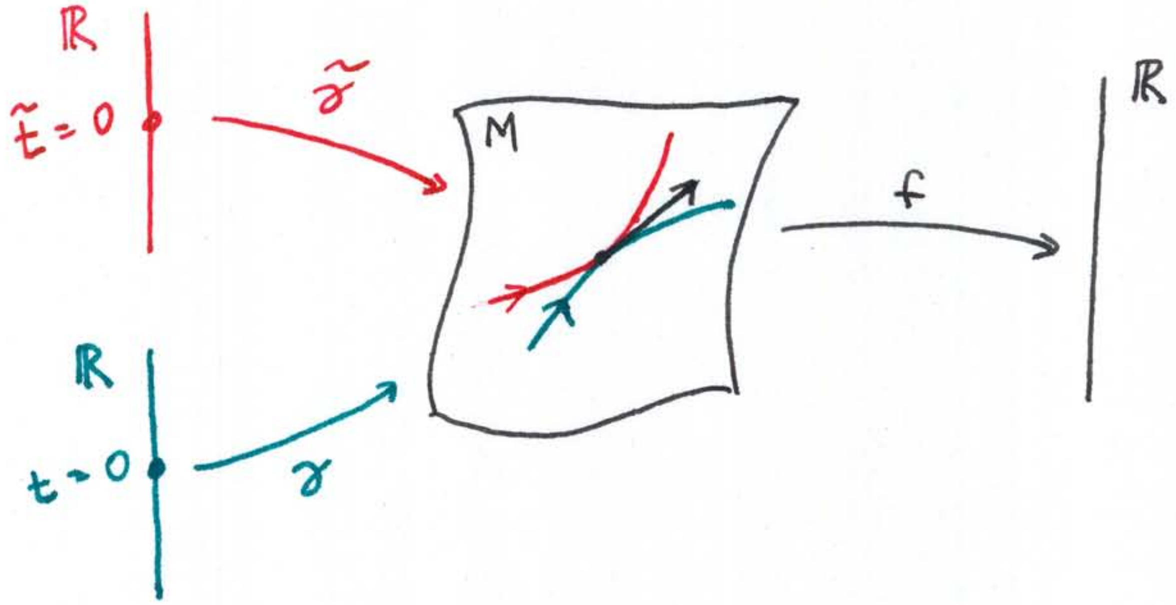
Tangent Vectors

The standard physical example of a tangent vector is the <u>velocity</u> of a particle at a point of its trajectory.

In mathematical language, let $\gamma : \mathbb{R} \to M$ be a <u>parameterized curve</u>. Then $\dot{\gamma}(t)$ should be a tangent vector to M at $\gamma(t)$.



But many curves will have the same velocity at a point:



Q: How do we describe the different tangent vectors without redundancy?

A: Derivative operators.

Let $f: M \to \mathbb{R}$ be any smooth function, and calculate

$$\dot{\gamma}(\overset{0}{\cancel{t}})(f) := \frac{d}{dt}(f \circ \gamma)\Big|_{t=0}$$

$$\dot{\tilde{\gamma}}(\overset{0}{\cancel{t}})(f) := \frac{d}{d\tilde{t}}(f \circ \tilde{\gamma})\Big|_{\tilde{t}=0}$$

equal if velocities identical.

Definition: A <u>tangent vector</u> V at a point $p \in M$ maps smooth functions f on M to numbers $V(f)$ such that:

- V is <u>linear</u>: $V(f+g) = V(f) + V(g)$
- V is <u>Leibniz</u>: $V(fg) = f(p)\,V(g) + g(p)\,V(f)$

<u>first-order</u> derivatives; p = <u>base point</u>

- V <u>annihilates constants</u>: $V(c) = 0$.

Note: These criteria imply linearity in the usual sense:

$$V(c_1 f_1 + c_2 f_2) = V(c_1 f_1) + V(c_2 f_2)$$

$$= c_1(p)\,V(f_1) + f_1(p)\,\cancel{V(c_1)}^{0}$$

$$+ c_2(p)\,V(f_2) \;(\text{o.k.}) + f_2(p)\,\cancel{V(c_2)}^{0}$$

$$= c_1\,V(f_1) + c_2\,V(f_2)$$

Tangent Space

The set of all tangent vectors with a given base point $p \in M$ is a vector space:

$$(\alpha_1 V_1 + \alpha_2 V_2)(f) :=$$
$$\alpha_1 V_1(f) + \alpha_2 V_2(f)$$

Need to show this is linear, Leibniz and annihilates constants.

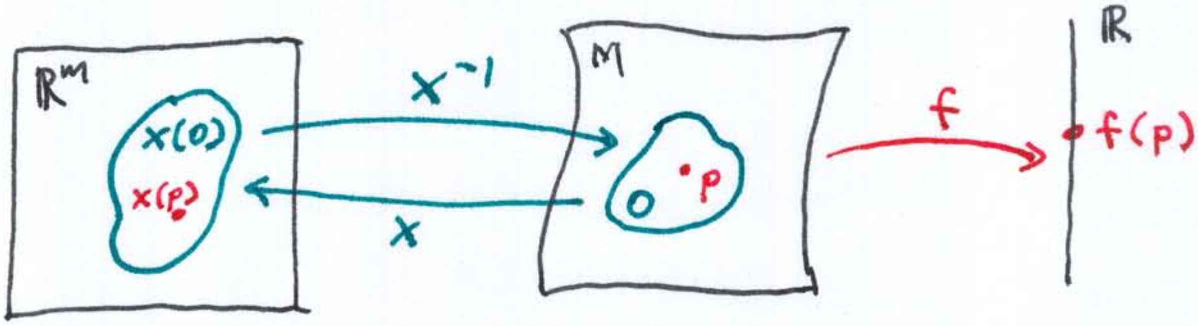
$$(v_1 + v_2)(f+g) :=$$
$$:= V_1(f+g) + V_2(f+g)$$
$$= V_1(f) + V_1(g) + V_2(f) + V_2(g)$$
$$= (V_1 + V_2)(f) + (V_1 + V_2)(g)$$

$$(V_1 + V_2)(fg) := V_1(fg) + V_2(fg)$$
$$= f(p)\, V_1(g) + g(p)\, V_1(\overset{f}{\cancel{g}})$$
$$+ f(p)\, V_2(g) + g(p)\, V_2(f)$$

(etc.) $$= f(p)\,(V_1 + V_2)(g) + g(p)\,(V_1 + V_2)(f)$$

Coordinate Basis

Every coordinate chart (x, O) at a point $p \in O \subset M$ defines a basis in the tangent space $T_p M$ at p.



$$\partial_\alpha (f) := \frac{\partial}{\partial x^\alpha} (f \circ x^{-1}) \Big|_{x(p)}$$

abstract tangent vector at p.

$f_x(x^1, \ldots, x^m)$

ordinary partial derivative

Theorem: (Taylor)

Remainder

$$f_x(x^1, \ldots, x^m) = f_x(0, \ldots, 0) + \sum_\alpha \frac{\partial f_x}{\partial x^\alpha}(0, \ldots, 0)\, x^\alpha + R(x^1, \ldots, x^m)$$

Now, we can write

$$f = f \circ x^{-1} \circ x = f_x \circ x$$

$$= f(p) + \partial_\alpha(f)\, x^\alpha + R$$

$$V(f) = \cancelto{0}{V(f(p))} + V(\partial_\alpha(f)\, x^\alpha) + \cancelto{0}{V(R)}$$

$$= \partial_\alpha(f) \cdot V(x^\alpha)$$

basis vectors acting on f.

components V^α of V in the ∂_α basis.

Thus, the tangent space $T_p M$ is <u>naturally</u> an m-dimensional vector space!