Lecture 11

Curvature (Finally!)

Curvature

Last time: Torsion

$$2\nabla_{[a}\nabla_{b]} f = -T_{ab}{}^{c}\nabla_c f$$

Now apply a similar commutator of derivatives to a tensor:

Key Question: functional linearity

$$\nabla_a\nabla_b(f\omega_c) = \nabla_a(f\nabla_b\omega_c + \omega_c\nabla_b f)$$

$$= f\nabla_a\nabla_b\omega_c + 2\nabla_{(a}f\cdot\nabla_{b)}\omega_c$$

$$+\omega_c\nabla_a\nabla_b f$$

$$\Rightarrow 2\nabla_{[a}\nabla_{b]}(f\omega_c) = 2f\nabla_{[a}\nabla_{b]}\omega_c$$

$$+2\omega_c\nabla_{[a}\nabla_{b]} f$$

$$-\omega_c T_{ab}{}^{d}\nabla_d f = -T_{ab}{}^{d}\left[\nabla_d(f\omega_c) - f\nabla_d\omega_c\right]$$

$$2\nabla_{[a}\nabla_{b]}(f\omega_c) + T_{ab}{}^{d}\nabla_d(f\omega_c)$$

$$= f\left[2\nabla_{[a}\nabla_{b]}\omega_c + T_{ab}{}^{d}\nabla_d\omega_c\right]$$

There is a tensor field $R_{abc}{}^{d}$ such that

$$2\nabla_{[a}\nabla_{b]}\omega_c + T_{ab}{}^{d}\nabla_d\omega_c = R_{abc}{}^{d}\omega_d$$

at every point of M for every co-vector field ω_c.

This is the curvature of the connection ∇_a.

What about other tensors?

$$\omega_c \nabla_a \nabla_b V^c = \nabla_a(\omega_c \nabla_b V^c) - \nabla_a\omega_c \cdot \nabla_b V^c$$

$$= \nabla_a\nabla_b(\omega_c V^c) - \nabla_a(V^c\nabla_b\omega_c)$$

$$- \nabla_a\omega_c \cdot \nabla_b V^c$$

$$= \nabla_a\nabla_b(\omega_c V^c) - 2\nabla_{(a}V^c\nabla_{b)}\omega_c$$

$$- V^c\nabla_a\nabla_b\omega_c$$

Now we anti-symmetrize:

$$2 w_c \nabla_{[a} \nabla_{b]} V^c = 2 \nabla_{[a} \nabla_{b]} (w_c V^c) - 2V^c \nabla_{[a} \nabla_{b]} w_c$$

$$= - T_{ab}{}^d \nabla_d (w_c V^c) - V^c (R_{abc}{}^d w_d - T_{ab}{}^d \nabla_d w_c)$$

$$= - w_c T_{ab}{}^d \nabla_d V^c - V^d R_{abd}{}^c w_c$$

$$2 \nabla_{[a} \nabla_{b]} V^c + T_{ab}{}^d \nabla_d V^c = - R_{abd}{}^c V^d$$

More generally, we find

$$2 \nabla_{[a} \nabla_{b]} S_{c_1 \cdots c_m}{}^{d_1 \cdots d_n} + T_{ab}{}^e \nabla_e S_{c_1 \cdots c_m}{}^{d_1 \cdots d_n}$$

$$= \sum_{i=1}^{m} R_{abc_i}{}^e S_{c_1 \cdots e \cdots c_m}{}^{d_1 \cdots d_n} - \sum_{j=1}^{n} R_{abe}{}^{d_j} S_{c_1 \cdots c_m}{}^{d_1 \cdots e \cdots d_n}$$

Geometrical Interpretation

$$0 = 2\nabla_{[a}\nabla_{b]} Z^d + T_{ab}{}^c \nabla_c Z^d + R_{abc}{}^d Z^c$$

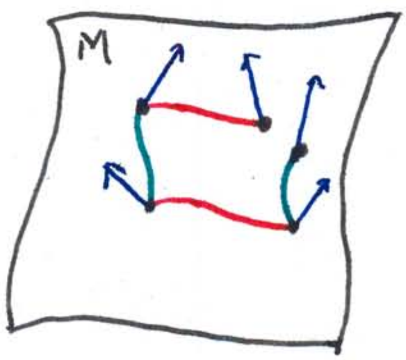
$$X^a Y^b \nabla_a \nabla_b Z^d =$$

$$= X^a \nabla_a (Y^b \nabla_b Z^d) - X^a \nabla_a Y^b \cdot \nabla_b Z^d$$

$$= \nabla_X \nabla_Y Z^d - \nabla_{\nabla_X Y} Z^d$$

$$0 = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y} Z$$

$$+ \nabla_{\nabla_Y X} Z + \nabla_{T(X,Y)} Z + R(X,Y,Z)$$



Transport Z and Y along X everywhere:

$$\nabla_X Y = \nabla_X Z = 0$$

Transport X and Z along Y <u>only at initial point</u>

$$\nabla_Y X \doteq \nabla_Y Z \doteq 0 \leftarrow$$ cannot take derivatives!

$$0 = \nabla_X \nabla_Y Z + \nabla_{T(X,Y)} Z + R(X,Y,Z)$$

Bianchi Identities

There are two ways to calculate

$$2\nabla_{[a}\nabla_b\nabla_{c]}f = \nabla_{[a}(2\nabla_b\nabla_{c]}f)$$

$$= -\nabla_{[a}(T_{bc]}{}^d\,\nabla_d f)$$

$$= -\nabla_{[a}T_{bc]}{}^d \cdot \nabla_d f - T_{[bc}{}^d\,\nabla_{a]}\nabla_d f$$

$$2\nabla_{[a}\nabla_b\nabla_{c]}f = 2\nabla_{[a}\nabla_b(\nabla_{c]}f)$$

$$= R_{[abc]}{}^d\,\nabla_d f - T_{[ab}{}^d\,\nabla_{|d|}\nabla_{c]}f$$

left out of anti-symmetrization

Equating these gives

$$(R_{[abc]}{}^d + \nabla_{[a}T_{bc]}{}^d)\,\nabla_d f$$

$$= -T_{[ab}{}^d(\nabla_{c]}\nabla_d f - \nabla_{|d|}\nabla_{c]}f)$$

$$= -T_{[ab}{}^d(-T_{c]d}{}^e\,\nabla_e f)$$

$$0 = (\underbrace{R_{[abc]}{}^d + \nabla_{[a}T_{bc]}{}^d - T_{[ab}{}^m\,T_{c]m}{}^d}\,)\,\nabla_d f$$

This tensor vanishes $\rightsquigarrow$ Bianchi <u>identity</u>

Similarly, there are two ways to calculate

$$2\nabla_{[a}\nabla_b\nabla_{c]}w_d = \nabla_{[a}\left(2\nabla_b\nabla_{c]}w_d\right)$$

$$= \nabla_{[a}\left(R_{bc]d}{}^e w_e - T_{bc]}{}^m \nabla_m w_d\right)$$

$$= \nabla_{[a}R_{bc]d}{}^e \cdot w_e + R_{[bc|d|}{}^e \nabla_{a]} w_e$$

$$- \nabla_{[a}T_{bc]}{}^m \cdot \nabla_m w_d - T_{[bc}{}^m \nabla_{a]}\nabla_m w_d$$

$$2\nabla_{[a}\nabla_b\nabla_{c]}w_d = 2\nabla_{[a}\nabla_b\left(\nabla_{c]}w_d\right)$$

$$= R_{[abc]}{}^e \nabla_e w_d + R_{[ab|d|}{}^e \nabla_{c]} w_e$$

$$- T_{[ab}{}^m \nabla_{|m|}\nabla_{c]} w_d$$

Combining these results gives

$$\nabla_{[a}R_{bc]d}{}^e \cdot w_e =$$

$$= \big(\underbrace{R_{[abc]}{}^m + \nabla_{[a}T_{bc]}{}^m}_{T_{[ab}{}^n T_{c]n}{}^m}\big)\nabla_m w_d$$

$$+ T_{[ab}{}^m\big(\underbrace{\nabla_{c]}\nabla_m w_d - \nabla_{|m|}\nabla_{c]} w_d}_{R_{c]md}{}^e w_e - T_{c]m}{}^n \nabla_n w_d}\big)$$

The first derivative terms cancel, leaving

$$\nabla_{[a} R_{bc]d}{}^{e} \cdot w_e = T_{[ab}{}^{m} R_{c]md}{}^{e} \cdot w_e$$

for <u>all</u> co-vector fields w_e.

We therefore have found two <u>Bianchi identities</u>

$$\nabla_{[a} T_{bc]}{}^{d} - T_{[ab}{}^{m} T_{c]m}{}^{d} + R_{[abc]}{}^{d} = 0$$

$$\nabla_{[a} R_{bc]d}{}^{e} - T_{[ab}{}^{m} R_{c]md}{}^{e} = 0$$

These hold <u>automatically</u> for <u>every</u> derivative operator ∇_a

Note that these become

$$R_{[abc]}{}^{d} = 0 \quad \text{and} \quad \nabla_{[a} R_{bc]d}{}^{e} = 0$$

when ∇_a is torsion-free.

Geodesics (Auto-Parallels)

A curve $\gamma(t)$ in a spacetime manifold M is non-accelerating when the spatial projection of the rate of change of its velocity vanishes in the instantaneously co-moving frame.

$$\text{“}\,\underbrace{\frac{d}{dt}}\, U^a \propto U^a\,\text{”}$$

What does this mean?

The velocity is defined by

$$U(f) := \frac{d}{dt}\big(f(\gamma(t))\big)$$

$$\rightsquigarrow \text{“}\, U = \frac{d}{dt} \,\text{”}$$

So, the non-acceleration condition can be written

$$U^a \nabla_a U^b = \alpha\, U^b$$

Affine Parameterization

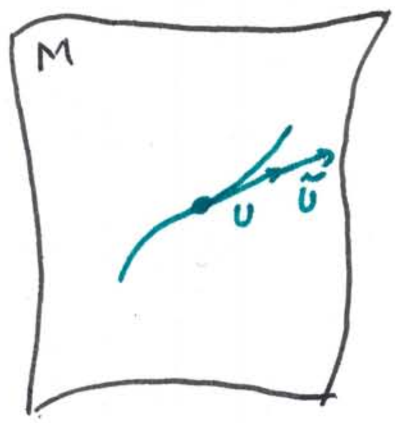
Suppose we <u>reparameterize</u> the curve γ, relabeling its points with "times"

$$t \mapsto \tilde{t} = \tilde{t}(t) \leftarrow \text{invertible } \mathbb{R} \to \mathbb{R}$$

We define a new curve

$$\tilde{\gamma}(\tilde{t}) := \gamma(t(\tilde{t}))$$

Same <u>locus</u> of points, but traversed at a different <u>rate</u>.



$\leadsto$ velocity <u>scaled</u>

$$\dot{\tilde{\gamma}}^a = \frac{d}{d\tilde{t}} = \frac{dt}{d\tilde{t}} \frac{d}{dt} = \rho \dot{\gamma}^a$$

reparameterization scaling

Reparameterize an auto-parallel:

$$\tilde{\dot\gamma}^a \nabla_a \tilde{\dot\gamma}^b = (\rho\dot\gamma^a)\nabla_a(\rho\dot\gamma^b)$$

$$= \rho\dot\gamma^a \nabla_a \rho \cdot \dot\gamma^b + \rho^2 \dot\gamma^a \nabla_a \dot\gamma^b$$

$$= \rho\dot\gamma^a \nabla_a \rho \cdot \dot\gamma^b + \rho^2 \alpha \dot\gamma^b$$

$$= \rho^2(\alpha + \dot\gamma^a \nabla_a \ln\rho)\dot\gamma^b$$

$$= \underbrace{\rho(\alpha + \dot\gamma^a \nabla_a \ln\rho)}_{\tilde\alpha}\tilde{\dot\gamma}^b$$

$\Rightarrow$ Auto-parallels are reparameterization invariant

$\Rightarrow$ Property of <u>curves</u>, not parameterized curves

Give auto-parallel with $\alpha \neq 0$, can solve

$$\dot\gamma^a \nabla_a \ln\rho = \frac{d}{dt}\ln\rho = -\alpha$$

so that $\tilde\alpha = 0$. (<u>affine parameter</u>)

The affine parameter is not unique:

$$\frac{d}{dt} \ln \rho = -\alpha$$

$$\Rightarrow \ln \rho = -\int^t \alpha(t')\,dt'$$

$$\Rightarrow \frac{d\tilde{t}}{dt} = \rho^{-1} = e^{\int^t \alpha(t')\,dt'}$$

$$\Rightarrow \tilde{t} = \int^t e^{\int^{t'} \alpha(t'')\,dt''}\,dt'$$

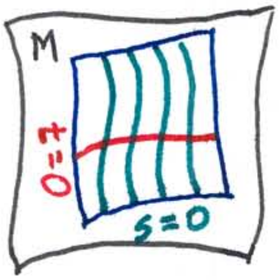
integration constants

$$\Rightarrow \tilde{t} \mapsto A\tilde{t} + B \quad \text{ambiguity}$$

<u>affine transformation</u> in $\mathbb{R}$

Geodesic Deviation

Let $C(s)$ be a smooth one-parameter family of geodesics:



For each s, $C(s)$ is an auto-parallel, and the curve varies smoothly with s.

⇝ smooth two-dimensional submanifold Σ of M, ruled by geodesics

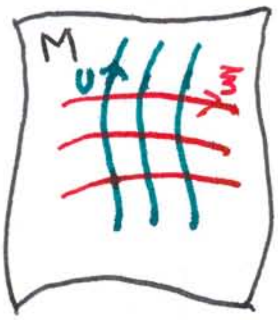
choose an affine parameter t on each $C(s)$ such that (t, s) form a smooth coordinate chart on Σ.

ambiguity: $(s, t) \mapsto (\tilde{s}, \tilde{t})$

$$= (s, A(s)\tilde{t} + B(s))$$

smooth functions of parameter s

Define vector fields on Σ



$$U(f) := \frac{\partial f}{\partial t}$$

$$\xi(f) := \frac{\partial f}{\partial s}$$

"points to corresponding point on nearby curve"

Note: $[U, \xi](f) = \frac{\partial}{\partial t}\frac{\partial f}{\partial s} - \frac{\partial}{\partial s}\frac{\partial f}{\partial t} = 0$

$\rightsquigarrow [U, \xi] = 0$

"relative velocity $= \dot{\xi}$"

$$\nabla_U \xi = [U, \xi]_\nabla + \nabla_\xi U$$

$$= \nabla_\xi U + T(U, \xi) + \underbrace{\cancel{[U, \xi]}}_{=0}$$

initial velocity difference

anomalous relative velocity

seems to vanish in Nature

$\rightsquigarrow$ torsion-free ∇_a

"relative acceleration = $\ddot{\xi}$"

(assuming no torsion)

$$\nabla_U \nabla_U \xi = \nabla_U (\nabla_\xi U)$$

$$= [\nabla_U, \nabla_\xi] U + \nabla_\xi(\cancel{\nabla_U U}) \quad {=0}$$

$$= -R(U, \xi, U) + \nabla_{\cancel{[U,\xi]}} U \quad {=0}$$

$$= R(\xi, U, U)$$

Note:

$$0 = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]_\nabla} Z + \nabla_{T(X,Y)} Z + R(X, Y, Z)$$

$$= [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z + R(X,Y,Z)$$

The relative acceleration of non-accelerating curves is given by the curvature of the connection.

Exercise: Calculate geodesic deviation <u>with</u> torsion.