Lecture 12

Riemannian Geometry

Calculating Curvature and Torsion

Suppose we know the curvature $\mathring{R}_{abc}{}^{d}$ and torsion $\mathring{T}_{ab}{}^{c}$ of a fiducial connection $\mathring{\nabla}_a$.

Let ∇_a be another connection:

$$(\nabla_a - \mathring{\nabla}_a)\,\omega_b = C_{ab}{}^{c}\,\omega_c$$

What are $T_{ab}{}^{c}$ and $R_{abc}{}^{d}$?

$$T_{ab}{}^{c}\,\nabla_c f = -2\,\nabla_{[a}\nabla_{b]} f$$

$$= -2\,\nabla_{[a}\mathring{\nabla}_{b]} f$$

$$= -2\,\mathring{\nabla}_{[a}\mathring{\nabla}_{b]} f - 2\,C_{[ab]}{}^{c}\,\mathring{\nabla}_c f$$

$$= \mathring{T}_{ab}{}^{c}\,\mathring{\nabla}_c f - 2\,C_{[ab]}{}^{c}\,\mathring{\nabla}_c f$$

$$= \left(\mathring{T}_{ab}{}^{c} - 2\,C_{[ab]}{}^{c}\right)\nabla_c f$$

$$\rightsquigarrow\quad T_{ab}{}^{c} = \mathring{T}_{ab}{}^{c} - 2\,C_{[ab]}{}^{c}$$

$$R_{abc}{}^{d}\, w_d = 2\nabla_{[a}\nabla_{b]}\, w_c + T_{ab}{}^{m}\,\nabla_m\, w_c$$

$$= 2\nabla_{[a}\left(\mathring{\nabla}_{b]}\, w_c + C_{b]c}{}^{d}\, w_d\right)$$

$$+ T_{ab}{}^{m}\left(\mathring{\nabla}_m\, w_c + C_{mc}{}^{d}\, w_d\right)$$

$$= 2\mathring{\nabla}_{[a}\left(\mathring{\nabla}_{b]}\, w_c + C_{b]c}{}^{d}\, w_d\right)$$

$$+ 2C_{[ab]}{}^{m}\left(\mathring{\nabla}_m\, w_c + C_{mc}{}^{d}\, w_d\right)$$

$$+ 2C_{[a|c|}{}^{m}\left(\mathring{\nabla}_{b]}\, w_m + C_{b]m}{}^{d}\, w_d\right)$$

$$+ T_{ab}{}^{m}\left(\mathring{\nabla}_m\, w_c + C_{mc}{}^{d}\, w_d\right)$$

$$= 2\mathring{\nabla}_{[a}\mathring{\nabla}_{b]}\, w_c + 2\mathring{\nabla}_{[a}C_{b]c}{}^{d}\cdot w_d$$

$$+ 2C_{[b|c|}{}^{d}\,\mathring{\nabla}_{a]}\, w_d$$

$$+ 2C_{[a|c|}{}^{m}\left(\mathring{\nabla}_{b]}\, w_m + C_{b]m}{}^{d}\, w_d\right)$$

$$+ \mathring{T}_{ab}{}^{m}\left(\mathring{\nabla}_m\, w_c + C_{mc}{}^{d}\, w_d\right)$$

$$= \mathring{R}_{abc}{}^{d}\, w_d - \mathring{T}_{ab}{}^{m}\,\mathring{\nabla}_m\, w_c$$

$$+ 2\left(\mathring{\nabla}_{[a}C_{b]c}{}^{d} + C_{[a|c|}{}^{m}C_{b]m}{}^{d}\right) w_d$$

$$+ \mathring{T}_{ab}{}^{m}\left(\mathring{\nabla}_m\, w_c + C_{mc}{}^{d}\, w_d\right)$$

Egad!

That's life!

So, the relationship between the curvature tensors is

$$R_{abc}{}^{d} = \mathring{R}_{abc}{}^{d} + 2\mathring{\nabla}_{[a} C_{b]c}{}^{d} + 2C_{[a|c|}{}^{m} C_{b]m}{}^{d} + \mathring{T}_{ab}{}^{m} C_{mc}{}^{d}$$

Note: A coordinate connection ∂_a is both <u>flat</u> ($\mathring{R}_{abc}{}^{d} = 0$) and <u>symmetric</u> ($\mathring{T}_{ab}{}^{c} = 0$).

$$\rightsquigarrow \quad T_{ab}{}^{c} = -2\,\Gamma_{[ab]}{}^{c}$$

$$R_{abc}{}^{d} = 2\left(\partial_{[a} \Gamma_{b]c}{}^{d} + \Gamma_{[a|c|}{}^{m} \Gamma_{b]m}{}^{d}\right)$$

This is one of several useful ways to calculate curvature in general relativity.

Basis Connections

Let b^a_α be a (local) basis of vector fields.

- i.e., the values of the n fields b^a_α at any point $p \in O \subset M$ form a basis for T_pM.
- There is a <u>unique</u> connection D_a such that

$$D_a b^b_\beta = 0 \qquad \beta = 1, \ldots, n$$

- D_a is <u>always</u> flat, $R_{abc}{}^d = 0$.
- D_a is also torsion-free, $T_{ab}{}^c = 0$, if and only if $b^a_\alpha = \partial^a_\alpha$ is a coordinate basis.

1) D_a <u>exists</u>:

<u>Define</u> $$D_a V^b := b^b_{\ \beta} \underbrace{D_a V^\beta}_{(dV^\beta)_a}$$

unambiguous

$$\leadsto D_a b^b_{\ \beta} := b^b_{\ \alpha} D_a \delta^\beta_{\ \alpha} = 0$$

$$\leadsto (D_a w_b) V^b = D_a (w_b V^b) - w_b D_a V^b$$

$$= D_a (w_\beta V^\beta) - \underbrace{w_b b^b_{\ \beta}}_{w_\beta} D_a V^\beta$$

$$= V^\beta D_a w_\beta = V^b b^\beta_{\ b} D_a w_\beta$$

$$\Longrightarrow \quad D_a w_b = b^\beta_{\ b} D_a w_\beta$$

$$\Longrightarrow \quad D_a b^\beta_{\ b} = 0.$$

2) D_a <u>unique</u>:

$$0 - 0 = (\tilde{D}_a - D_a) b^\beta_{\ b} = C_{ab}{}^c b^\beta_{\ c}$$

$$\Longrightarrow C_{ab}{}^c = 0 \Longrightarrow \tilde{D}_a = D_a .$$

3) D_a <u>flat</u>:

$$D_a D_b w_c = D_a \left(b_c^{\gamma} D_b w_{\gamma} \right)$$

$$= b_c^{\gamma} D_a D_b w_{\gamma}$$

$$\Rightarrow \quad 2 D_{[a} D_{b]} w_c = 2 b_c^{\gamma} D_{[a} D_{b]} w_{\gamma}$$

$$= - b_c^{\gamma} T_{ab}{}^{m} D_m w_{\gamma}$$

$$= - T_{ab}{}^{m} D_m \big(\underbrace{w_{\gamma} b_c^{\gamma}}_{w_c} \big)$$

$$\Rightarrow \quad 0 = 2 D_{[a} D_{b]} w_c + T_{ab}{}^{m} D_m w_c$$

$$:= R_{abc}{}^{d} w_d$$

4) D_a has torsion:

$$D_a D_b f = b_b^{\beta} D_a \left(b_{\beta}^{n} D_n f \right)$$

$$= b_a^{\alpha} b_b^{\beta} \cdot b_{\alpha}^{m} D_m \left(b_{\beta}^{n} D_n f \right)$$

$$= b_a^{\alpha} b_b^{\beta} b_{\alpha} \left(b_{\beta} (f) \right)$$

When we anti-symmetrize,

$$2 D_{[a} D_{b]} f = 2 b^{\alpha}_{[a} b^{\beta}_{b]} b_\alpha(b_\beta(f))$$

$$= 2 b^{[\alpha}_{a} b^{\beta]}_{b} b_\alpha(b_\beta(f))$$

$$= 2 b^{\alpha}_{a} b^{\beta}_{b} b_{[\alpha}(b_{\beta]}(f))$$

$$= b^{\alpha}_{a} b^{\beta}_{b} \left[b_\alpha(b_\beta(f)) - b_\beta(b_\alpha(f)) \right]$$

$$= b^{\alpha}_{a} b^{\beta}_{b} [b_\alpha, b_\beta] f$$

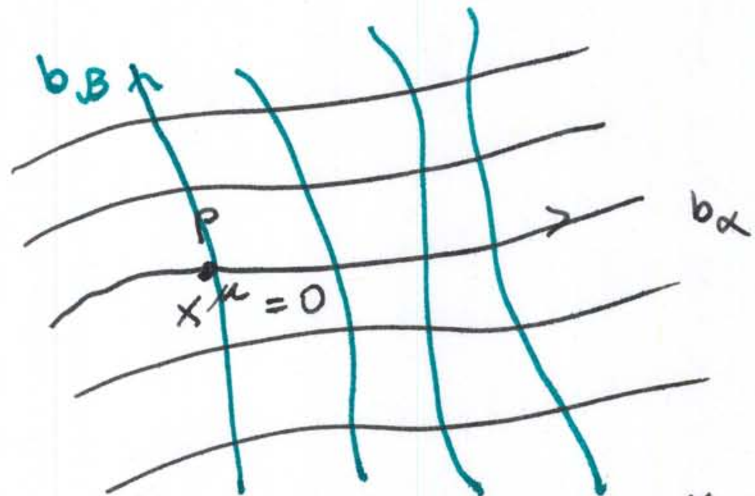
$$\Rightarrow \quad T_{ab}{}^{c} = - b^{\alpha}_{a} b^{\beta}_{b} [b_\alpha, b_\beta]^{c}$$

The torsion is given by the Lie brackets of basis fields.

These brackets vanish if and only if $b^{a}_{\alpha} = \partial^{a}_{\alpha}$.

Given b^a_α with

$$[b_\alpha, b_\beta] = 0$$



$$b_\alpha(x^\mu) = \delta^\mu_\alpha$$

Ricci Tensor

We define the Ricci tensor by contracting the curvature:

$$R_{ac} := R_{abc}{}^{b}$$

The Bianchi identities give

$$\begin{aligned} 0 &= R_{[abc]}{}^{b} \\ &= \tfrac{1}{3}\left(R_{abc}{}^{b} + R_{bca}{}^{b} + R_{cab}{}^{b}\right) \\ &= \tfrac{1}{3}\left(R_{ac} - R_{cba}{}^{b} - R_{acb}{}^{b}\right) \end{aligned}$$

$$\Rightarrow \quad R_{[ac]} = \tfrac{1}{2} R_{acb}{}^{b}$$

$$\begin{aligned} 0 &= \nabla_{[a} R_{bc]d}{}^{c} \\ &= \tfrac{1}{3}\left(2\nabla_{[a} R_{b]cd}{}^{c} + \nabla_{c} R_{abd}{}^{c}\right) \end{aligned}$$

$$\Rightarrow \quad \nabla_{[a} R_{b]c} = -\tfrac{1}{2} \nabla_{d} R_{abc}{}^{d}$$

Note: have assumed $T_{ab}{}^{c} = 0$.

Riemannian Geometry

A (pseudo-) Riemannian manifold is a pair (M, g_{ab})

- M is a smooth manifold
- g_{ab} is a non-degenerate, smooth, symmetric tensor field

- $g_{ab} X^a Y^b = 0$ for all Y^b in T_pM $\Rightarrow$ $X^a = 0$

  $\rightsquigarrow$ $g_{ab} : T_pM \to T_p^*M$ is invertible $\rightsquigarrow$ g^{ab}

$$g^{ab} g_{bc} = \delta^a_c$$

- $g_{ab} \doteq g_{ba}$ $\rightsquigarrow$ $g_{ab} X^a Y^b = X \cdot Y$

Note: Given X^a or w_a, define

$$X_b := X^a g_{ab} \qquad w^b := g^{ab} w_a$$

$$g_{ab} \underset{b^a_\alpha}{\rightsquigarrow} g_{\alpha\beta}$$

$$\downarrow$$

$$\left(g_{\alpha\beta} \right)^{-1} = g^{\alpha\beta}$$

$$\underset{b^a_\alpha}{\rightsquigarrow}$$

$$g^{ab} := g^{\alpha\beta}\, b^a_\alpha\, b^b_\beta$$

$$\downarrow$$

$$g^{ab} g_{bc} = \delta^a_c$$

The Metric Connection

There is a <u>unique</u> torsion-free connection ∇_a on any Riemannian manifold (M, g_{ab}) with

$$\nabla_a g_{bc} = 0 \longleftarrow \text{metric (compatible)}$$

1) ∇_a <u>exists</u>:

Let $\mathring{\nabla}_a$ be any fiducial torsion-free connection (e.g., ∂_a)

- $T_{ab}{}^c = \mathring{T}_{ab}{}^c - 2C_{[ab]}{}^c$

$$\Rightarrow C_{[ab]}{}^c = 0$$

- $(\nabla_a - \mathring{\nabla}_a) g_{bc} = C_{ab}{}^m g_{mc} + C_{ac}{}^m g_{bm}$

$$= C_{abc} + C_{acb}$$

$$= -\mathring{\nabla}_a g_{bc}$$

So, we have two equations:

$$C_{abc} = C_{bac}$$

$$C_{abc} = -\mathring{\nabla}_a g_{bc} - C_{acb}$$

Can we solve them simultaneously?

$$C_{abc} = -\mathring{\nabla}_a g_{bc} - C_{cab}$$

$$= -\mathring{\nabla}_a g_{bc} - (-\mathring{\nabla}_c g_{ab} - C_{cba})$$

$$= -\mathring{\nabla}_a g_{bc} + \mathring{\nabla}_c g_{ab} + C_{bca}$$

$$= -\mathring{\nabla}_a g_{bc} + \mathring{\nabla}_c g_{ab} \qquad C_{abc}$$

$$- \mathring{\nabla}_b g_{ca} - C_{bac}$$

$$\Rightarrow 2\, C_{abc} = \mathring{\nabla}_c g_{ab} - 2\, \mathring{\nabla}_{(a} g_{b)c}$$

$$C_{ab}{}^{c} = -\tfrac{1}{2} g^{cd} \left(2\, \mathring{\nabla}_{(a} g_{b)d} - \mathring{\nabla}_d g_{ab} \right)$$

↑

(sign conventions)

2) ∇_a is <u>unique</u>:

Suppose $\mathring{\nabla}_a$ is a torsion-free metric-compatible connection:

$$2C_{abc} = \mathring{\nabla}_c g_{ab} - 2\mathring{\nabla}_{(a} g_{b)c} = 0$$

$$\Rightarrow \nabla_a = \mathring{\nabla}_a$$

Note: Whenever ∇_a is metric-compatible, we have

$$\nabla_a X_b = \nabla_a (g_{bc} X^c) = g_{bc} \nabla_a X^c$$

$\leadsto$ Raising/lowering indices <u>commutes</u> with the covariant derivative.

<u>only</u> metric connection.

Riemann Curvature

The **Riemann Tensor** is the curvature of the symmetric metric connection ∇_a.

It has extra features:

$$0 = 2\nabla_{[a}\nabla_{b]} g_{cd}$$

$$= R_{abc}{}^{m} g_{md} + R_{abd}{}^{m} g_{cm}$$

$$= 2R_{ab(cd)}$$

$$\Rightarrow R_{[ab]cd} = R_{abcd} = R_{ab[cd]}$$

can't even define this without g_{ab}!

With no torsion, the Bianchi identities become

$$R_{[abc]d} = 0 \quad \text{and} \quad \nabla_{[a}R_{bc]de} = 0$$

The first Bianchi identity gives

$$R_{abcd} + R_{bcad} + R_{cabd} = 0$$

Combining this with the new anti-symmetry gives

$$R_{abcd} - R_{cdab} =$$

$$= - R_{bcad} - R_{cabd}$$

$$+ R_{dacb} + R_{acdb}$$

$$= R_{bcda} - R_{dabc} \leftarrow \text{shift all indices left}$$

$$= R_{cdab} - R_{abcd}$$

$$\Rightarrow \quad R_{abcd} = R_{cdab}$$

Note: This result is <u>not equivalent</u> to $R_{[abc]d} = 0$.

One must also require $R_{[abcd]} = 0$. (exercise)