Lecture 14

The Newtonian Limit

The perturbation in the curvature is given by:

$$2\nabla_{[a}\nabla_{b]} w_c = R_{abc}{}^d w_d$$

$$\Rightarrow \dot{R}_{abc}{}^d w_d = \cancel{2\dot{\nabla}_{[ab]}{}^m \nabla_m w_c}$$

$$+ 2\dot{\nabla}_{[a|c|}{}^m \nabla_{b]} w_m + 2\nabla_{[a}\left(\dot{\nabla}_{b]c}{}^d w_d\right)$$

$$= 2\dot{\nabla}_{[a|c|}{}^m \nabla_{b]} w_m + 2\dot{\nabla}_{[b|c|}{}^d \nabla_{a]} w_d$$

$$+ 2 w_d \nabla_{[a}\dot{\nabla}_{b]c}{}^d$$

$$\Rightarrow \dot{R}_{abc}{}^d = 2\nabla_{[a}\dot{\nabla}_{b]c}{}^d$$

$$\Rightarrow \dot{R}_{ac} = 2\nabla_{[a}\dot{\nabla}_{b]c}{}^b$$

$$= 2 g^{bd}\nabla_{[a}\dot{\nabla}_{b]cd}$$

$$\dot{\nabla}_{bcd} = -\tfrac{1}{2}\left(2\nabla_{(b}\dot{g}_{c)d} - \nabla_d \dot{g}_{bc}\right)$$

$$= -\tfrac{1}{2}\left(\nabla_b \dot{g}_{cd} + 2\nabla_{[c}\dot{g}_{d]b}\right)$$

Putting these results together,

$$\dot{R}_{ac} = -g^{bd}\,\nabla_{[a}\left(\nabla_{b]}\dot{g}_{cd} + 2\nabla_{[c}\dot{g}_{d]b]}\right)$$

$$= -\tfrac{1}{2} g^{bd}\Big(R_{abc}{}^{m}\dot{g}_{md} + R_{abd}{}^{m}\dot{g}_{cm} + \nabla_a\nabla_c\dot{g}_{db} - \nabla_a\nabla_d\dot{g}_{cb} - \nabla_b\nabla_c\dot{g}_{da} + \nabla_b\nabla_d\dot{g}_{ca}\Big)$$

$$= -\nabla_c\nabla_b\,\dot{g}_{da} - R_{bcd}{}^{m}\dot{g}_{ma} - R_{bca}{}^{m}\dot{g}_{dm}$$

$$\dot{R}_{ac} = -\tfrac{1}{2}\Big(\underline{R_a{}^b{}_c{}^m\dot{g}_{mb}} - \underline{\underline{R_a{}^m\dot{g}_{cm}}} + \nabla_a\nabla_c\dot{g} - \underline{\underline{\underline{\nabla_a\nabla^b\dot{g}_{cb}}}} - \underline{\underline{\underline{\nabla_c\nabla^b\dot{g}_{ba}}}} - \underline{\underline{R_c{}^m\dot{g}_{ma}}} - \underline{R^b{}_{ca}{}^m\dot{g}_{bm}} + \nabla_b\nabla^b\dot{g}_{ca}\Big)$$

$$\dot{g} := \dot{g}_m{}^m$$

$$\dot{R}_{ac} = -\tfrac{1}{2}\Big(\nabla_b\nabla^b\dot{g}_{ac} - 2\nabla_{(a}\nabla^b\dot{g}_{c)b} + \nabla_a\nabla_c\dot{g} - 2R_{(a}{}^b\dot{g}_{c)b} + 2R_a{}^b{}_c{}^d\dot{g}_{bd}\Big)$$

The perturbation of the Einstein tensor is given by

$$G_{ab} = R_{ab} - \tfrac{1}{2} R g_{ab}$$
$$= \left(\delta^m_a \delta^n_b - \tfrac{1}{2} g^{mn} g_{ab} \right) R_{mn}$$

Note that

$$g^{ab} g_{bc} = \delta^a_c$$

$$\Rightarrow \frac{d}{d\lambda} g^{ab} \cdot g_{bc} + g^{ab} \dot{g}_{bc} = 0$$

$$\Rightarrow \frac{d}{d\lambda} g^{ab} = - g^{am} \dot{g}_{mn} g^{nc} = - \dot{g}^{ab}$$

we need to be careful with signs. Our convention: $\dot{g}$ refers to the <u>covariant</u> metric perturbation. Indices are raised using the background metric.

Accordingly, we find

$$\dot{G}_{ab} = \left(\delta_a^m \delta_b^n - \tfrac{1}{2} g^{mn} g_{ab}\right) \dot{R}_{mn}$$
$$- \tfrac{1}{2}\left(- \dot{g}^{mn} g_{ab} + g^{mn} \dot{g}_{ab}\right) R_{mn}$$

$$= -\tfrac{1}{2}\left(\delta_a^m \delta_b^n - \tfrac{1}{2} g^{mn} g_{ab}\right)$$
$$\times \left(\nabla_c \nabla^c \dot{g}_{mn} - 2 \nabla_{(m} \nabla^c \dot{g}_{n)c} + \nabla_m \nabla_n \dot{g}\right.$$
$$\left. - 2 R_{(m}{}^c \dot{g}_{n)c} + 2 R_m{}^c{}_n{}^d \dot{g}_{cd}\right)$$
$$+ \tfrac{1}{2}\left(\dot{g}^{mn} g_{ab} - g^{mn} \dot{g}_{ab}\right) R_{mn}$$

$$= -\tfrac{1}{2} \nabla_c \nabla^c \left(\dot{g}_{ab} - \tfrac{1}{2} \dot{g}\, g_{ab}\right)$$
$$+ \left(\delta_a^m \delta_b^n - \tfrac{1}{2} g^{mn} g_{ab}\right) \nabla_{(m} \nabla^c \left(\dot{g}_{n)c} - \tfrac{1}{2} \dot{g}\, g_{n)c}\right)$$
$$+ \left(\delta_a^m \delta_b^n - \tfrac{1}{2} \cancel{g^{mn}} g_{ab}\right)\left(R_{(m}{}^c \dot{g}_{n)c}\right.$$
$$\left. - R_m{}^c{}_n{}^d \dot{g}_{cd}\right)$$
$$+ \tfrac{1}{2}\left(\dot{g}^{mn} g_{ab} - g^{mn} \dot{g}_{ab}\right) R_{mn}$$

Define the "trace-reversed" metric perturbation

$$h_{ab} := \dot{g}_{ab} - \tfrac{1}{2}\dot{g}\, g_{ab}$$

Then the Einstein perturbation takes the "simple" form

$$\begin{aligned}
\dot{G}_{ab} = &-\tfrac{1}{2}\nabla_c \nabla^c h_{ab} \\
&+ \left(\delta^m_a \delta^n_b - \tfrac{1}{2} g^{mn} g_{ab}\right) \nabla_{(m}\nabla^c h_{n)c} \\
&+ R_{(a}{}^c h_{b)c} - R_a{}^c{}_b{}^d h_{cd} \\
&- \tfrac{1}{2} R\, h_{ab} + \tfrac{1}{2} g_{ab} R^{cd} h_{cd}
\end{aligned}$$

$$\boxed{\begin{aligned}
\dot{G}_{ab} = &-\tfrac{1}{2}\nabla_c\nabla^c h_{ab} + G_{(a}{}^c h_{b)c} \\
&+ \left(\delta^m_a \delta^n_b - \tfrac{1}{2} g^{mn} g_{ab}\right) \\
&\quad \times \left(\nabla_{(m}\nabla^c h_{n)c} - R_m{}^c{}_n{}^d h_{cd}\right)
\end{aligned}}$$

Post-Minkowski Gravity

If all gravitational sources in a spacetime region are weak, the background geometry is

$$\mathring{g}_{ab} = \eta_{ab} \qquad \mathring{\nabla}_a = \partial_a$$

The first-order field equation then becomes

$$\dot{G}_{ab} = -\tfrac{1}{2}\partial_c \partial^c h_{ab} + \partial_{(a}\partial^c h_{b)c} - \tfrac{1}{2}\partial^c\partial^d h_{cd} \cdot \eta_{ab}$$

$$= 8\pi \dot{T}_{ab}$$

The field operator acting on $h_{\cdot\cdot}$ here is very similar to the Maxwell operator:

$$\partial^a \dot{G}_{ab} = \frac{d}{d\lambda}\Big(\underbrace{\nabla^a G_{ab}}_{=0}\Big) - \text{"}\,\dot{\nabla} \times \underbrace{G}_{=0}\text{"}$$

$$\frac{d}{d\lambda}\left(\nabla_a G^{ab}\right)\Big|_{\lambda=0}$$

$$= -\dot{\nabla}_{am}{}^{a}\,\overset{\circ}{G}{}^{mb} - \dot{\nabla}_{am}{}^{b}\,\overset{\circ}{G}{}^{am}$$

$$+ \overset{\circ}{\nabla}_a\,\dot{G}^{ab}$$

$$\Rightarrow \overset{\circ}{\nabla}_a\,\dot{T}^{ab} = 0$$

$$\dot{G}^{ab} = 8\pi\,\cancel{G}\,\dot{T}^{ab}$$

Like the Maxwell equation, the post-Minkowski equation has a gauge ambiguity:

Diffeomorphism: $\dot{g}_{ab} = 2\partial_{(a}\xi_{b)}$

$$\rightsquigarrow \quad h_{ab} = 2\partial_{(a}\xi_{b)} - \partial_c \xi^c \eta_{ab}$$

Put this "pure gauge" field into the field equation:

$$-\tfrac{1}{2}\partial_c\partial^c h_{ab} + \partial_{(a}\partial^c h_{b)c} - \tfrac{1}{2}\eta_{ab}\partial^c\partial^d h_{cd}$$

$$= \underline{-\partial_c\partial^c\partial_{(a}\xi_{b)}} + \underline{\underline{\tfrac{1}{2}\eta_{ab}\partial_c\partial^c\partial_d\xi^d}}$$

$$+ \underline{\partial_{(a}\partial^c\partial_{b)}\xi_c} + \underline{\partial_{(a}\partial^c\partial_{|c|}\xi_{b)}}$$

$$- \underline{\underline{\partial_{(a}\partial^c(\eta_{b)c}\partial_d\xi^d)}}$$

$$- \underline{\underline{\underline{\tfrac{1}{2}\eta_{ab}(2\partial^c\partial^d\partial_c\xi_d - \partial^c\partial_c\partial_d\xi^d)}}}$$

$$= 0$$

As with the Maxwell equations, we can use this gauge freedom to our advantage:

$$\partial^a (h_{ab} + 2\partial_{(a}\xi_{b)})$$

$$= \partial^a h_{ab} + \partial^a \partial_a \xi_b + \partial^a \partial_b \xi_a$$

$\rightsquigarrow$ solve

$$\partial^a \partial_a \xi_b + \partial_b \partial^a \xi_a = -\partial^a h_{ab}$$

$\Rightarrow$ $\tilde{h}_{ab} := h_{ab} + 2\partial_{(a}\xi_{b)}$

is in <u>de Donder gauge</u>

$$\partial^a \tilde{h}_{ab} = 0$$

This field <u>still</u> solves the post-Minkowski field equation, but many terms vanish:

$$-\frac{1}{2}\partial_c \partial^c \tilde{h}_{ab} = 8\pi \dot{T}_{ab}$$

$$\Box A^a - \nabla^a \nabla_b A^b = -4\pi j^a$$

Lorentz gauge: $\nabla_b A^b = 0$

gauge trans: $A_a \to \tilde{A}_a = A_a + \nabla_a \psi$

$$0 = \nabla_a A^a + \nabla_a \nabla^a \psi$$

$$\Rightarrow \quad \Box \psi = -\nabla_a A^a$$

$$\Box \tilde{A}_a - \nabla^a \underbrace{\cancel{(\nabla_b \tilde{A}^b)}}_{0} = -4\pi j^a$$

$$\boxed{\begin{gathered} \Box \tilde{A}_a = -4\pi j^a \\ \nabla_a \tilde{A}^a = 0 \end{gathered}}$$

$$-\frac{1}{2}\Box h_{ab} + \partial_{(a}\partial^{c} h_{b)c} - \frac{1}{2}\eta_{ab}\partial^{c}\partial^{d} h_{cd}$$

$$= 8\pi \dot{T}_{ab}$$

<u>de Donder</u>: $\partial^{c} h_{bc} = 0$

$$\tilde{h}_{ab} = h_{ab} + 2\partial_{(a}\xi_{b)}$$

$$\partial^{a}\tilde{h}_{ab} = \partial^{a} h_{ab} + \partial^{a}\partial_{a}\xi_{b} + \partial_{b}\partial^{a}\xi_{a}$$

$$\Big\downarrow$$

$$\xi_{b}$$

$$\Box \xi_{b} + \partial_{b}\partial^{a}\xi_{b} = -\partial^{a} h_{ab}$$

$$\boxed{\Box \tilde{h}_{ab} = -16\pi \dot{T}_{ab}}$$