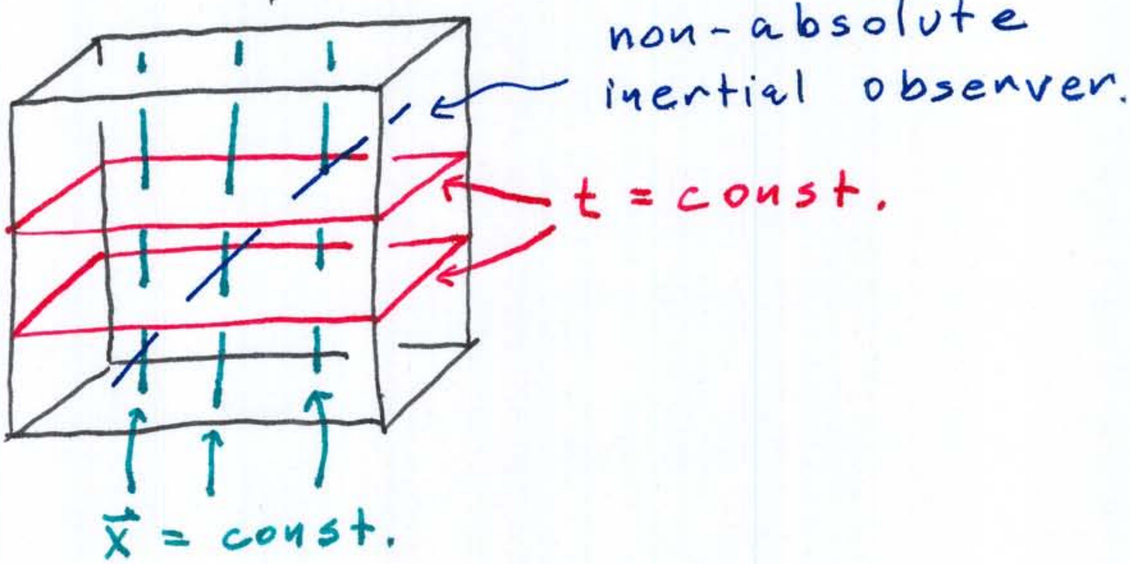
Lecture 15

The Newtonian Limit

Newtonian Space-Time

Newtonian physics has a Universal time variable t and a preferred state of absolute rest, occupied by inertial observers with 4-velocity t^a.



The Newtonian limit of general relativity arises when sources are (a) weak and (b) move slowly enough that spacetime has approximately this structure. (t, t^a)

The Newtonian Limit

The Newtonian gravitational field is given by solving the Poisson equation

$$\Delta \Phi = 4\pi \rho$$

for the gravitational potential Φ in terms of the mass density ρ.

To recover this limit from general relativity, we must assume that

- source speeds are slow $v \ll c$
- we are close to the source: $r \ll cT$ ← dynamical time scale

(no retardation)

In these limits, to order unity in the source speed, we may write

$$T_{ab} \approx \rho\, t_a t_b$$

mass density — absolute rest

When we are close enough to ignore retardation, we may also neglect time-derivatives in the de-Donder-gauge post-Minkowski field equation:

$$\Box h_{ab} \approx \Delta h_{ab} = -16\pi T_{ab}$$

Thus, we have the first-order <u>post-Newtonian</u> field equation

$$\Delta h_{ab} = -16\pi \rho\, t_a t_b$$

The vector field t^a satisfies

$$\partial_a t^b = 0$$

and so commutes with the Laplacian on the Newtonian spatial slices ($t = \text{const.}$)

$$\Rightarrow \quad h_{ab} = -4\Phi t_a t_b$$

$$\text{with} \quad \Delta \Phi = 4\pi\rho \leftarrow \text{Newtonian potential}$$

The metric perturbation is

$$h_{ab} = \dot{g}_{ab} - \tfrac{1}{2}\eta_{ab}\dot{g}$$

$$\Rightarrow h = \dot{g} - \tfrac{1}{2}\cdot 4\dot{g} = -\dot{g}$$

$$\Rightarrow \dot{g}_{ab} = h_{ab} + \tfrac{1}{2}\eta_{ab}\dot{g}$$

$$= h_{ab} - \tfrac{1}{2}\eta_{ab} h$$

$$= -4\Phi t_a t_b - \tfrac{1}{2}\eta_{ab}\cdot -4\Phi \overset{\overset{-1}{\downarrow}}{t^c t_c}$$

$$= -2\Phi(2t_a t_b + \eta_{ab})$$

To leading order, the physical metric is therefore

$$g_{ab} = \eta_{ab} + \dot{g}_{ab}$$

$$= (-t_a t_b + \sigma_{ab}) - 2\Phi(t_a t_b + \sigma_{ab})$$

$$= -(1+2\Phi)\, t_a t_b + (1-2\Phi)\, \sigma_{ab}$$

spatial metric on Newtonian slices.

Note: We have built the perturbation parameter λ into the potential here:

$$\Phi = \frac{\text{potential energy}}{\text{unit mass}} \sim \frac{E}{M} \sim c^2$$

$$ds^2 = -(c^2 + 2\Phi)\,dt^2 + \left(1 - \frac{2\Phi}{c^2}\right)(dx^2 + dy^2 + dz^2)$$

For the field outside a star, e.g., we must have $\frac{GM_*}{c^2 R_*} \ll 1$.

Motion of Test Bodies

Working in the background "Newtonian" inertial coordinates, the geodesic equation is

$$u^a \nabla_a u^b = 0$$

$$\rightsquigarrow \quad \frac{d^2 x^\beta}{d\tau^2} - \Gamma_{\alpha\gamma}{}^{\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\gamma}{d\tau} = 0$$

For slowly moving test masses, we may write, to leading order,

$$\frac{dx^\alpha}{d\tau} \sim t^\alpha \quad \text{and} \quad d\tau \sim dt$$

Thus, we have

$$\frac{d^2 x^\beta}{dt^2} = \Gamma_{00}{}^{\beta} = -\frac{1}{2} g^{\beta\alpha}\left(2\partial_{(0} g_{0)\alpha} - \partial_\alpha g_{00}\right)$$

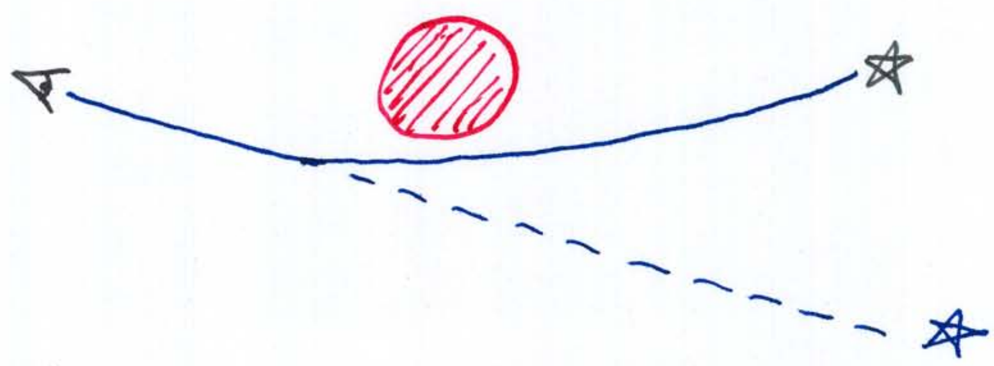
$$= \frac{1}{2} \partial^\beta \left(-c^2 - 2\Phi\right)$$

$$= -\partial^\beta \Phi \quad \leftarrow \quad \boxed{\ddot{\vec{x}} = -\vec{\nabla} \Phi}$$

Deflection of Light

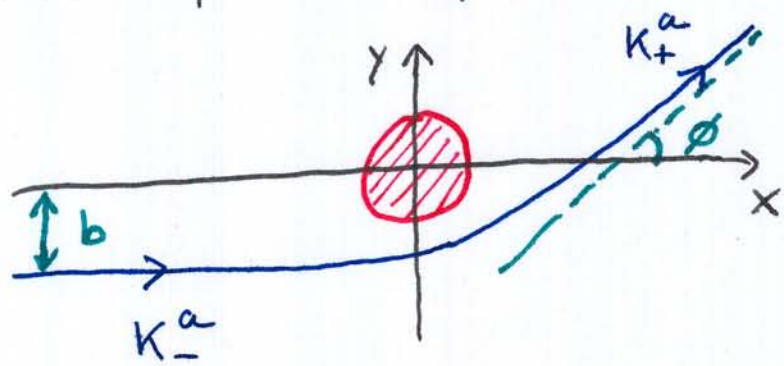
Light is not affected by gravity in Newtonian physics, but it follows geodesics even in post-Minkowskian gravity.

Q: How does the gravitational field of the sun affect light from distant stars?



A: We must repeat the previous calculation for a <u>null</u> geodesic.

Set up the problem as follows:



$\tan\phi = \dfrac{K_+^y}{K_+^x}$

Let λ be an affine parameter along the null geodesic:

$$K^\alpha = \frac{dx^\alpha}{d\lambda} \quad \text{and} \quad \frac{dK^\alpha}{d\lambda} = \Gamma_{\mu\nu}{}^\alpha K^\mu K^\nu$$

$$\rightsquigarrow \frac{dK^y}{d\lambda} = \Gamma_{tt}{}^y K^t K^t + \Gamma_{xx}{}^y K^x K^x$$

(Note: $\Gamma_{\mu\nu}{}^a$ is already first-order, so we can use the zeroth-order approximants $K^x = K^t$.)

$$\Gamma_{tt}{}^{y} = -\tfrac{1}{2} g^{yy} \left(2\partial_{(t}\, g_{t)y} - \partial_y\, g_{tt} \right)$$

$$= \tfrac{1}{2}\partial^{y} \left(-c^2 - 2\Phi \right) = -\partial^{y}\Phi$$

$$\Gamma_{xx}{}^{y} = -\tfrac{1}{2} g^{yy} \left(2\partial_{(x}\, g_{x)y} - \partial_y\, g_{xx} \right)$$

$$= \tfrac{1}{2}\partial^{y} \left(1 - 2\Phi \right) = -\partial^{y}\Phi$$

$$\rightsquigarrow \frac{dK^{y}}{d\lambda} = -2\partial^{y}\Phi \cdot K^{x} K^{x}$$

Now let's take the potential

$\Phi = -\frac{M}{r}$ outside the sun:

$$\frac{\partial\Phi}{\partial y} = \frac{My}{r^3} \cong \frac{-Mb}{(x^2+b^2)^{3/2}}$$

$$\rightsquigarrow \frac{dK^{y}}{d\lambda} = \frac{2Mb \cdot K^{x}}{(x^2+b^2)^{3/2}} \frac{dx}{d\lambda}$$

$$\Rightarrow \Delta K^{y} = 2MbK^{x}\, \Delta \frac{x}{b^2\sqrt{x^2+b^2}}$$

$$= \frac{4M}{b} K^{x}$$

This gives the deflection

$$\phi \cong \tan\phi \cong \frac{4M}{b}$$

For example, for the sun, we can calculate

$$\frac{4M_\odot}{R_\odot} = \frac{4GM_\odot}{c^2 R_\odot}$$

(cgs)

$$= \frac{4(6.67\times10^{-8})(1.99\times10^{33})}{(3.00\times10^{10})^2(6.96\times10^{10})}$$

$$= 8.48\times10^{-6} \leftarrow \text{(radians)}$$

$$= 1.75''$$

Thus, we find the deflection

$$\phi_\odot = (1.75'')\,\frac{R_\odot}{b}$$

observed by Eddington.

Gravitational Radiation

Suppose we continue to work with weak, slow-motion sources, but look at large distances where we cannot ignore retardation effects:

$$\Box h_{ab} = -16\pi T_{ab}$$

$$h_{\alpha\beta}(t, \vec{x}) = 4\int \frac{T_{\alpha\beta}(t-|\vec{x}-\vec{y}|, \vec{y})}{|\vec{x}-\vec{y}|}\, d^3y$$

background inertial components

integral over past light cone

Retarded scalar Green function:

$$G_y(x) = -\frac{1}{2\pi}\,\theta_{t_y}(t_x)\,\delta(\|x-y\|^2)$$

$$= -\frac{1}{4\pi}\,\frac{\delta(t_y - t_x + |\vec{x}-\vec{y}|)}{|\vec{x}-\vec{y}|}$$

Fourier transform in t

$$\tilde{h}_{\alpha\beta}(\omega, \vec{x}) = 4 \int \frac{\tilde{T}_{\alpha\beta}(\omega, \vec{y})}{|\vec{x} - \vec{y}|} e^{i\omega|\vec{x}-\vec{y}|} d^3y$$

Look in the "wave zone"

$\omega R \gg 1$

$$\tilde{h}_{\alpha\beta}(\omega, \vec{x}) = 4 \frac{e^{i\omega R}}{R} \int \tilde{T}_{\alpha\beta}(\omega, \vec{y}) \, d^3y$$