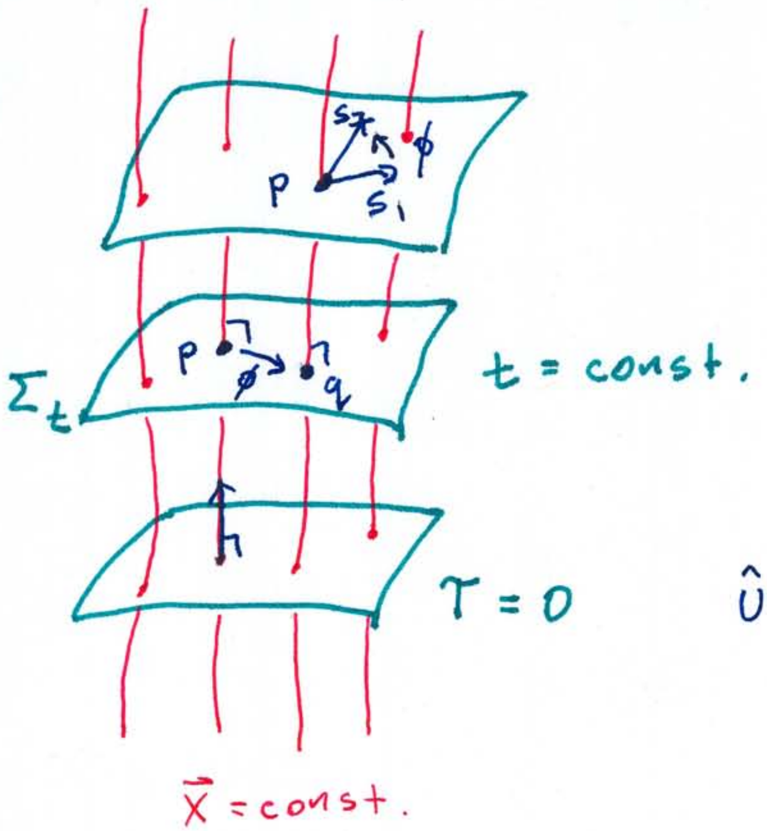
Homogeneity and Isotropy



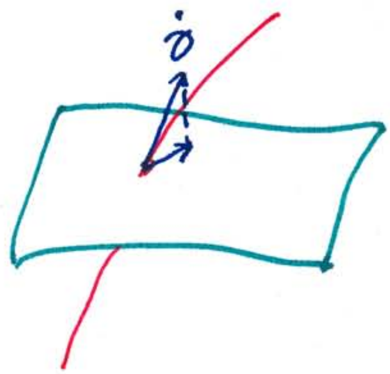
$\hat{u}^a \nabla_a \tau = 1$

Spacetime is

- <u>spatially homogeneous</u> if it is <u>foliated</u> by a one-parameter family of spacelike surfaces Σ_t such that for each pair of points $p, q \in \Sigma_t$, there is a spacetime isometry $\phi(p) = q$.

If spacetime is <u>both</u> homogeneous <u>and</u> isotropic, then

$$\underline{\dot{\gamma}^a_x \perp \Sigma_t}$$



In addition, each Σ_t must be <u>maximally symmetric</u>.

$$g_{ab} = -\hat{U}_a \hat{U}_b + q_{ab}(\tau)$$

unit tangent to $\gamma(\tau)$

maximally symmetric spatial metric.

Spacetime is

- <u>isotropic</u> if it is <u>ruled</u> by a family of timelike curves γ_x such that, for each $p \in \gamma_x$ and each pair of unit vectors $s^a_{1,2} \in T_p M$ that are $\perp$ to $\dot{\gamma}^a$, there is a spacetime isometry ϕ with $\phi(p) = p$ and

$$\phi : s^a_1 \to s^a_2$$

Consider just space.

Riemann tensor on space:

$$r_{ab}{}^{cd}\, w_{cd} = r(w)_{ab}$$

(Riemann maps 2-forms to 2-forms linearly.)

$\rightsquigarrow$ eigen-2-forms w_{ab}

$\rightsquigarrow$ λ_i, w^i_{ab} ← preferred 2-forms.

$\rightsquigarrow$ $v^i_a := \frac{1}{2}\varepsilon_a{}^{bc}\, w^i_{bc}$

↖ preferred vectors.

Isotropy implies that <u>every</u> 2-form w_{ab} is an eigen-form with the <u>same</u> eigenvalue K.

$$r_{ab}{}^{cd} = K\, \delta^c_{[a}\, \delta^d_{b]}$$

Homogeneity demands that K is constant on each Σ_t

$\rightsquigarrow$ Σ_t is a <u>space of constant curvature</u>.

I) Positive curvature: $K > 0$

This is the geometry of a 3-sphere in $\mathbb{E}^4$:

$$w^2 + x^2 + y^2 + z^2 = K^{-1} \leftarrow \text{extrinsic}$$

$$ds^2 = K^{-1}\left(d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$

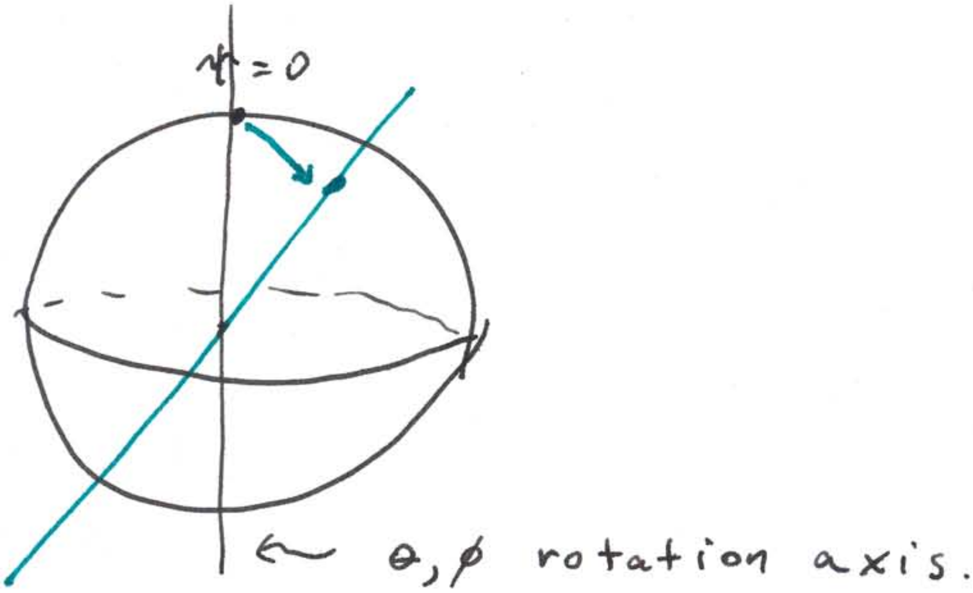
II) Negative curvature: $K < 0$

$$-w^2 + x^2 + y^2 + z^2 = K^{-1}$$

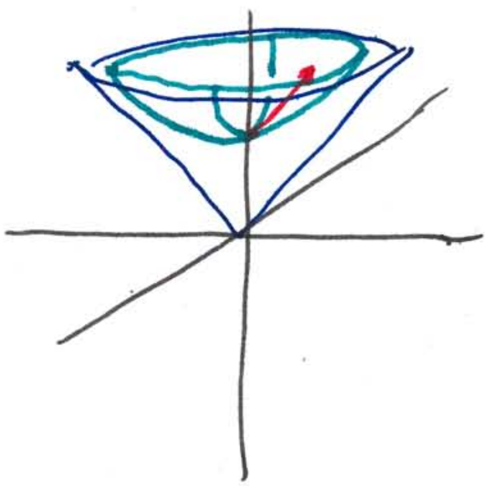
$$ds^2 = -K^{-1}\left(d\psi^2 + \sinh^2\psi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right)$$

III) Zero curvature

$$ds^2 = d\psi^2 + \psi^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$



$$ds^2 = d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$



$$ds^2 = d\psi^2 + \sinh^2\psi \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

The spacetime metric must be of the form

$$ds^2 = -d\tau^2 + a^2(\tau)\, h_0$$

maximally symmetric spatial metric of <u>unit</u> radius

$K = +1, -1, 0$

$\rightsquigarrow$ metric has <u>one</u> undetermined component $a(\tau)$, depending on only time.

Robertson - Walker metric

$K > 0$:

$$h = d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

define $\quad r := \sin\psi$

$$dr = \cos\psi\, d\psi = \sqrt{1-r^2}\, d\psi$$

$$h = \frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$K < 0$:

$$h = d\psi^2 + \sinh^2\psi\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$r := \sinh\psi$

$$dr = \cosh\psi\, d\psi = \sqrt{1+r^2}\, d\psi$$

$$h = \frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$h = \frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$\uparrow$

$\pm 1, 0$

$\mathbb{E}^3 : \quad dx^2 + dy^2 + dz^2 = {}^3ds^2$

$\downarrow$

$S^2 : \quad ds^2 = dx^2 + dy^2 + dz^2$

$$x\,dx + y\,dy + z\,dz = 0$$

$${}^3ds^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$\uparrow$

$r = \text{const.}$

$${}^2ds^2 = r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$