Last time

$$ds^2 = -d\tau^2 + a^2(\tau)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

$K = \pm 1, 0$

Because of symmetry, the Einstein tensor must have the form:

$$G_{ab} = G_{00} U_a U_b + G_{**} q_{ab}$$

In the same way, the stress-energy must have the form

$$T_{ab} = \rho U_a U_b + P q_{ab}$$

$$\leadsto \boxed{\begin{aligned} a^{-2}(\dot{a}^2 + K) &= \frac{8\pi}{3}\rho \\ a^{-1}\ddot{a} &= -\frac{4\pi}{3}(\rho + 3P) \end{aligned}}$$

$$G_{ab} = 8\pi T_{ab}$$

Big Bang

For physically realistic matter,

$$\rho > 0 \quad \text{and} \quad P \geq 0$$

$$a^{-2}(\dot{a}^2 + k) = \frac{8\pi}{3}\rho \qquad a^{-1}\ddot{a} = -\frac{4\pi}{3}(\rho + 3P)$$

$$\downarrow$$

$$\ddot{a} < 0$$

$$\Downarrow$$

$$\dot{a}(\tau_0) = 0 \Rightarrow k = +1 \longleftarrow \dot{a} \neq 0$$

Since $\ddot{a} < 0$ and $\dot{a} > 0$ right now,

τ = past time. $\quad\tau_0$

$$\dot{a}(\tau) > \dot{a}(\tau_0)$$

$$\Rightarrow \int_{\tau}^{\tau_0} \dot{a}(\bar{\tau})\, d\bar{\tau} = a(\tau_0) - a(\tau)$$

$$> \dot{a}(\tau_0)(\tau_0 - \tau)$$

$$\tau_{min}: \quad \tau_0 - \tau_{min} = \frac{a(\tau_0)}{\dot{a}(\tau_0)}$$

Hubble Expansion

The distance between any two preferred observers scales with $a(\tau)$

$$\frac{\dot{D}}{D} = \frac{\dot{a}}{a} =: H \leftarrow$$ Hubble expansion constant.

The age of the universe τ_0 is bounded above by the current value of the Hubble constant:

$$\tau_0 < \frac{1}{H(\tau_0)}$$

$\rightsquigarrow$ beginning of time.

Fate of the Universe

$$\dot{a}^2 + K = \frac{8\pi}{3}\rho a^2 \qquad \frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P)$$

$$\Rightarrow 2\dot{a}\ddot{a} = \frac{8\pi}{3}\left(\dot{\rho} a^2 + 2\rho a \dot{a}\right)$$

$$= 2\dot{a} \cdot - a \frac{4\pi}{3}(\rho + 3P)$$

$$0 = \frac{8\pi}{3}\left(\dot{\rho} a^2 + a\dot{a}(3\rho + 3P)\right)$$

$$\dot{\rho} a^2 = -3a\dot{a}(\rho + P)$$

$$\frac{\dot{\rho}}{\rho + P} = -3\frac{\dot{a}}{a}$$

$$\rho + P \geq \rho \Rightarrow \frac{\dot{\rho}}{\rho} \leq -3\frac{\dot{a}}{a}$$

$$\ln \frac{\rho(\tau_0)}{\rho(\tau)} \leq -3 \ln \frac{a(\tau_0)}{a(\tau)}$$

$$\rho(\tau_0)\, a^3(\tau_0) \leq \rho(\tau)\, a^3(\tau)$$

$\tau > \tau_0$

$$\ln \frac{\rho}{\rho_0} \le -3 \ln \frac{a}{a_0}$$

$$\rho a^3 \le \rho_0 a_0^3$$

1) $K = +1$

$$\dot{a}^2 + 1 = \frac{8\pi}{3} \rho a^2 \le \frac{8\pi}{3} \frac{\rho_0 a_0^3}{a}$$

$$\Downarrow$$

there is a time τ_c $\quad$ $\ddot{a}(\tau_c) < 0$

with $\dot{a}(\tau_c) = 0$

$$a(\tau_c) \le \frac{8\pi}{3} \rho a_0^3$$

2) $K = 0, -1$

$$\dot{a}^2 + K = \frac{8\pi}{3} \rho a^2 > 0$$

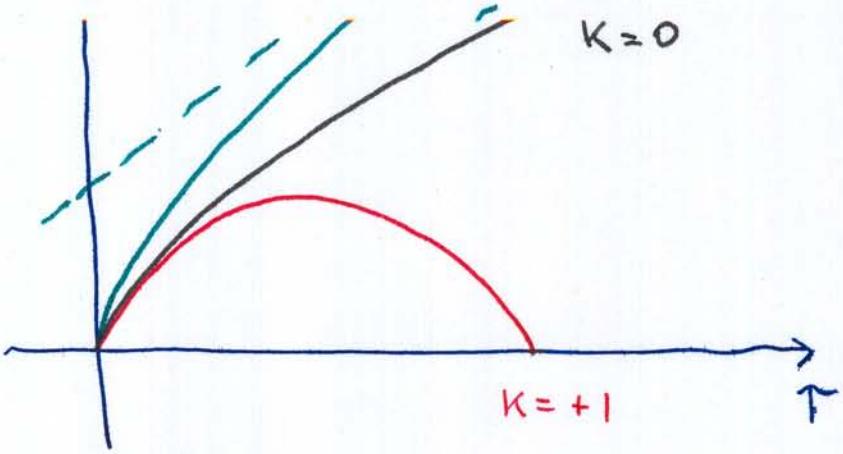
$$\dot{a}^2 > -K \geq 0 \quad \Rightarrow \quad \dot{a} > 0$$

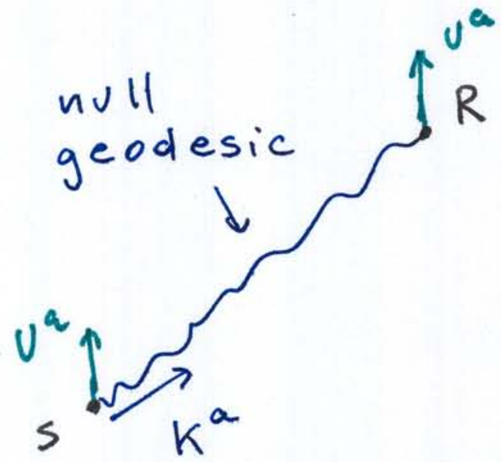
$\Rightarrow$ perpetual expansion

$$0 < \frac{8\pi}{3} \rho a^2 \leq \frac{8\pi}{3} \frac{\rho_0 a_0^3}{a}$$

$$\frac{8\pi}{3} \rho a^2 = \dot{a}^2 + K$$

$$-K < \dot{a}^2 \leq -K + \frac{8\pi}{3} \rho_0 a_0^3 \, a^{-1}$$


K=0
K=+1



$$\frac{\omega_R}{\omega_S} = \frac{K^a U_a(\tau_R)}{K^a U_a(\tau_S)}$$