Geometry Exercises II 

1 a) For functions, we have

$$\mathcal{L}_{\bar{v}} \mathcal{L}_{\bar{w}} f = \mathcal{L}_{\bar{v}} (\bar{w}(f)) = \bar{v}(\bar{w}(f))$$

$$\Rightarrow [\mathcal{L}_{\bar{v}}, \mathcal{L}_{\bar{w}}](f) = (\bar{v}\bar{w} - \bar{w}\bar{v})(f) = [\bar{v}, \bar{w}](f)$$

$$= \mathcal{L}_{[\bar{v}, \bar{w}]} f$$

For vector fields,

$$\mathcal{L}_{\bar{v}} \mathcal{L}_{\bar{w}} \bar{X} = \mathcal{L}_{\bar{v}} [\bar{w}, \bar{X}] = [\bar{v}, [\bar{w}, \bar{X}]]$$

$$\Rightarrow [\mathcal{L}_{\bar{v}}, \mathcal{L}_{\bar{w}}] \bar{X} = [\bar{v}, [\bar{w}, \bar{X}]] - [\bar{w}, [\bar{v}, \bar{X}]]$$

$$= [\bar{v}, [\bar{w}, \bar{X}]] + [\bar{w}, [\bar{X}, \bar{v}]]$$

$$= -[\bar{X}, [\bar{v}, \bar{w}]] = [[\bar{v}, \bar{w}], \bar{X}]$$

$$= \mathcal{L}_{[\bar{v}, \bar{w}]} \bar{X}$$

The key step here has used the Jacobi identity.

b) Since $\mathcal{L}_{\bar{v}}(f) = \bar{v}(f)$, we have done the scalar part of this in assignment 1. For vectors,

$$[[\mathcal{L}_{\bar{X}}, \mathcal{L}_{\bar{Y}}], \mathcal{L}_{\bar{Z}}] \bar{V} = [\mathcal{L}_{[\bar{X}, \bar{Y}]}, \mathcal{L}_{\bar{Z}}] \bar{V} = \mathcal{L}_{[[\bar{X}, \bar{Y}], \bar{Z}]} \bar{V}$$

Since $\mathcal{L}_{\bar{X}}$ is linear in $\bar{X}$, the result follows once again from the Jacobi identity.

2 a) We have

$$\mathcal{L}_{\bar{V}}(f\bar{U}) := [\bar{V}, f\bar{U}] := \bar{V}(f\bar{U}) - f\bar{U}\bar{V}$$

$$= \bar{V}(f)\cdot\bar{U} + f\,\bar{V}\bar{U} - f\bar{U}\bar{V}$$

$$=: \mathcal{L}_{\bar{V}} f \cdot \bar{U} + f\,[\bar{V}, \bar{U}] = \bar{U}\,\mathcal{L}_{\bar{V}} f + f\,\mathcal{L}_{\bar{V}}\bar{U}$$

b) Here, we have

$$\mathcal{L}_{\bar{V}}\bar{U} = [\bar{V}, \bar{U}] = [V^i\bar{e}_i, U^j\bar{e}_j]$$

$$= V^i[\bar{e}_i, U^j\bar{e}_j] - U^j\bar{e}_j(V^i)\,\bar{e}_i$$

$$= V^iU^j[\bar{e}_i, \bar{e}_j] + V^i\bar{e}_i(U^j)\,\bar{e}_j - U^j\bar{e}_j(V^i)\,\bar{e}_i$$

The result follows immediately.

c) This follows from (2.7):

$$(\mathcal{L}_{\bar{V}}\bar{W})^i = \delta^j_1 \frac{\partial}{\partial x^j} W^i - W^j \frac{\partial}{\partial x^j}\delta^i_1 = \frac{\partial W^i}{\partial x^1}$$

We have noted that the only non-zero component of $\bar{V}$ here is $V^1 = 1$.

3 By the Leibniz property

$$(\mathcal{L}_{\bar{V}}\tilde{\omega})\circ\bar{W} = \mathcal{L}_{\bar{V}}(\tilde{\omega}\circ\bar{W}) - \tilde{\omega}\circ\mathcal{L}_{\bar{V}}\bar{W}$$

$$= \bar{V}(\tilde{\omega}(\bar{W})) - \tilde{\omega}([\bar{V}, \bar{W}])$$

Expressing things in components gives

$$(\mathcal{L}_{\bar{V}} \tilde{w})_i w^i = V^j \partial_j (w_i w^i) - w_i (V^j \partial_j w^i - w^j \partial_j V^i)$$

$$= w^i V^j \partial_j w_i + w^j w_i \partial_j V^i$$

$$= w^i (V^j \partial_j w_i + w_j \partial_i V^j)$$

The result follows because $\bar{w}$ is arbitrary.

4 From (6.17) we have

$$\mathcal{L}_X Z_{bc} = X^m \partial_m Z_{bc} + Z_{mc} \partial_b X^m + Z_{bm} \partial_c X^m$$

$$\mathcal{L}_X (Y^a Z_{bc}) = X^m \partial_m (Y^a Z_{bc}) - (Y^m Z_{bc}) \partial_m X^a + (Y^a Z_{mc}) \partial_b X^m + (Y^a Z_{bm}) \partial_c X^m$$

$$= Z_{bc} (X^m \partial_m Y^a - Y^m \partial_m X^a) + Y^a (X^m \partial_m Z_{bc} + Z_{mc} \partial_b X^m + Z_{bm} \partial_c X^m)$$

The Leibniz property is confirmed in the second equality.

5 a) This is obvious. The denominator $1 \pm z$ vanishes only at $z = \mp 1$. These points must therefore be excluded from the corresponding chart.

b) First, we must calculate $\psi_s^{-1}$:

$$(u, v) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \rightsquigarrow u^2 + v^2 = \frac{x^2 + y^2}{(1-z)^2} = \frac{1+z}{1-z}$$

In the last equality, we have recalled that $x^2+y^2+z^2=1$. Inverting this, we find

$$(1-z)(u^2+v^2) = 1+z \quad \rightsquigarrow \quad z = \frac{u^2+v^2-1}{u^2+v^2+1}$$

$$\rightsquigarrow \quad 1-z = \frac{2}{u^2+v^2+1}$$

$$\Rightarrow (x,y,z) = ((1-z)u, (1-z)v, z) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

We now compose this with $\psi_N$:

$$\psi_N(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right) =: (u', v')$$

$$1+z = 1 + \frac{u^2+v^2-1}{u^2+v^2+1} = \frac{2(u^2+v^2)}{u^2+v^2+1}$$

$$\Rightarrow (u', v') = \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)$$

This map should be defined for all points except the two poles. Thus, $u'=v'=0$ (south pole) and $u=v=0$ (north pole) must be excluded. Except at the origin, the function $\psi_N \circ \psi_S^{-1}(\vec{r}) = \frac{\vec{r}}{|\vec{r}|^2}$ in two dimensions is clearly smooth.

c) This function also has a smooth inverse, which happens to be the same function from $\mathbb{R}^2$ to $\mathbb{R}^2$. Therefore, all of the overlap functions are smooth, and $S^2$ is a manifold.

6 a) We need to show that $\varsigma_1$ takes the same value for every $(z^1, z^2)$ on a given "line." But this is immediate: $(\alpha z^1)/(\alpha z^2) = z^1/z^2$.

b) $\xi_1$ is defined unless $z^2 = 0$, meaning except on the line $[1, 0]$. Similarly, $\xi_2$ is defined except on $[0, 1]$.

c) If $w = \xi_1([z^1, z^2]) = z^1/z^2$, then

$$[w, 1] = [z^1/z^2, 1] = [z^1, z^2]$$

In the second equality, we have scaled both $z$'s by $z^2$, which doesn't change the line. The proof for $\xi_2$ is identical.

d) Both charts are defined unless $z^1 = 0$ or $z^2 = 0$. Throwing out these points, we have

$$\xi_2 \circ \xi_1^{-1}(w) = \xi_2([w, 1]) = \frac{1}{w}$$

Since $\xi_1^{-1}(0) = [0, 1]$ is excluded, this mapping from $\mathbb{C}$ to $\mathbb{C}$ is analytic, and therefore smooth. The proof for $\xi_1 \circ \xi_2^{-1}$ is identical.

e) All overlap functions are analytic. This is done!

7 a) We have

$$[1+z, x+iy] = \left[\tfrac{x-iy}{1+z}(1+z), \tfrac{x-iy}{1+z}(x+iy)\right]$$

$$= \left[x-iy, \frac{x^2+y^2}{1+z}\right]$$

$$= \left[x-iy, \frac{1-z^2}{1+z}\right] = [x-iy, 1-z]$$

This is the result.

b) From the first expression, we have

$$\frac{x+iy}{1+z} = \frac{z^2}{z^1} \Rightarrow \left|\frac{z^2}{z^1}\right|^2 = \frac{x^2+y^2}{(1+z)^2} = \frac{1-z}{1+z}$$

$$\Rightarrow z = \frac{1-|z^2/z^1|^2}{1+|z^2/z^1|^2} = \frac{|z^1|^2-|z^2|^2}{|z^1|^2+|z^2|^2}$$

$$\Rightarrow x+iy = (1+z)\frac{z^2}{z^1} = \frac{2|z^1|^2}{|z^1|^2+|z^2|^2}\frac{z^2}{z^1} = \frac{2\bar{z}^1 z^2}{|z^1|^2+|z^2|^2}$$

Taking real and imaginary parts gives the result.

c) We have, for example

$$\zeta_1 \circ \phi \circ \psi_N^{-1}(u,v) = \zeta_1 \circ \phi\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{-u^2+v^2-1}{u^2+v^2+1}\right)$$

$$= \zeta_1\left(\left[1-\frac{u^2+v^2-1}{u^2+v^2+1}, \frac{2(u+iv)}{u^2+v^2+1}\right]\right)$$

$$= \zeta_1\left(\left[\frac{2}{u^2+v^2+1}, \frac{2}{u^2+v^2+1}(u+iv)\right]\right) = u+iv$$

This is obviously smooth with smooth inverse. The other coordinate maps can be found by composing with the transition functions calculated above:

$$\zeta_2 \circ \phi \circ \psi_N^{-1}(u,v) = \frac{1}{u+iv}$$

$$\zeta_1 \circ \phi \circ \psi_S^{-1}(u,v) = \frac{1}{u-iv}$$

$$\zeta_2 \circ \phi \circ \psi_S^{-1}(u,v) = u-iv$$

d) Since all of these maps are smooth wherever both charts are defined, the manifolds are diffeomorphic in the same sense that $\mathbb{C}$ can be identified with $\mathbb{R}^2$.