Geometry Exercises III

1 We have

$$\nabla_c (X_a Y^a) = Y^a \nabla_c X_a + X_a \nabla_c Y^a$$

$$\Rightarrow \partial_c (X_a Y^a) = Y^a \nabla_c X_a + X_a (\partial_c Y^a + \Gamma^a{}_{bc} Y^b)$$

$$= X_a \partial_c Y^a + Y^a \partial_c X_a$$

$$\Rightarrow \quad Y^a \nabla_c X_a = Y^a \partial_c X_a - Y^a \Gamma^b{}_{ac} X_b$$

We have interchanged the indices $a \leftrightarrow b$ in the last equality, and (6.26) follows because Y^a is arbitrary.

2 The tangent to the curve in the $\bar{s}$ parameterization is

$$\bar{X} = \frac{d}{d\bar{s}} = \frac{ds}{d\bar{s}} \frac{d}{ds} = \left(\frac{d\bar{s}}{ds}\right)^{-1} \frac{d}{ds}$$

There fore, we have

$$\nabla_{\bar{X}} \bar{X} = \left(\frac{d\bar{s}}{ds}\right)^{-1} \nabla_X \left[\left(\frac{d\bar{s}}{ds}\right)^{-1} X \right]$$

$$= \left(\frac{d\bar{s}}{ds}\right)^{-1} \left[X \nabla_X \left(\frac{d\bar{s}}{ds}\right)^{-1} + \left(\frac{d\bar{s}}{ds}\right)^{-1} \nabla_X X \right]$$

$$= \left(\frac{d\bar{s}}{ds}\right)^{-1} \frac{d}{ds} \left(\frac{d\bar{s}}{ds}\right)^{-1} \cdot X = -\left(\frac{d\bar{s}}{ds}\right)^{-2} \frac{d^2\bar{s}}{ds^2} \bar{X}$$

The right side vanishes iff $\frac{d^2\bar{s}}{ds^2} = 0$, which implies that $\bar{s} = \alpha s + \beta$.

3 Choosing an arbitrary Y^b, we have

$$2\nabla_{[c}\nabla_{d]}(X^a{}_b Y^b) = R^a{}_{ecd} X^e{}_b Y^b$$

$$= 2\nabla_{[c}\left(Y^b \nabla_{d]} X^a{}_b + \nabla_{d]} Y^b \cdot X^a{}_b\right)$$

$$= 2Y^b \nabla_{[c}\nabla_{d]} X^a{}_b + 2\nabla_{[c} Y^b \cdot \nabla_{d]} X^a{}_b$$

$$+ 2\nabla_{[d} Y^b \cdot \nabla_{c]} X^a{}_b + 2X^a{}_b \nabla_{[c}\nabla_{d]} Y^b$$

$$= 2Y^b \nabla_{[c}\nabla_{d]} X^a{}_b + X^a{}_b R^b{}_{ecd} Y^e$$

$$\Rightarrow Y^b \cdot 2\nabla_{[c}\nabla_{d]} X^a{}_b = Y^b\left(R^a{}_{ecd} X^e{}_b - R^e{}_{bcd} X^a{}_e\right)$$

The result follows.

4 We have

$$\nabla_X \nabla_Y Z^a = X^c \nabla_c (Y^d \nabla_d Z^a)$$

$$= X^c \nabla_c Y^d \cdot \nabla_d Z^a + X^c Y^d \nabla_c \nabla_d Z^a$$

Antisymmetrizing on X and Y gives

$$[\nabla_X, \nabla_Y] Z^a = (X^c \nabla_c Y^d - Y^c \nabla_c X^d)\nabla_d Z^a + (X^c Y^d - Y^c X^d)\nabla_c \nabla_d Z^a$$

$$= [X, Y]^d \nabla_d Z^a + 2X^c Y^d \nabla_{[c}\nabla_{d]} Z^a$$

$$= \nabla_{[X,Y]} Z^a + X^c Y^d R^a{}_{bcd} Z^b.$$

5 These are all diagonal metrics, so

$$(x,y,z): \; g_{ab} \sim \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \rightsquigarrow g^{ab} \sim \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \rightsquigarrow g = 1$$

$$(R,\phi,z): \; g_{ab} \sim \begin{pmatrix} 1 & & \\ & R^2 & \\ & & 1 \end{pmatrix} \rightsquigarrow g^{ab} \sim \begin{pmatrix} 1 & & \\ & R^{-2} & \\ & & 1 \end{pmatrix} \rightsquigarrow g = R^2$$

$$(r,\theta,\phi): \; g_{ab} \sim \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2\theta \end{pmatrix} \rightsquigarrow g^{ab} \sim \begin{pmatrix} 1 & & \\ & r^{-2} & \\ & & r^{-2}\sin^{-2}\theta \end{pmatrix}$$

$$\rightsquigarrow g = r^4 \sin^2\theta$$

6 The only non-trivial metric component in this case is $g_{\phi\phi} = R^2$, so the only nonvanishing Christoffel symbols must have two ϕ indices and one R:

$$\{\phi\phi, R\} = -\tfrac{1}{2}\partial_R g_{\phi\phi} = -R$$

$$\{\phi R, \phi\} = \{R\phi, \phi\} = \tfrac{1}{2}\partial_R g_{\phi\phi} = R$$

$$\rightsquigarrow \left\{ {R \atop \phi\phi} \right\} = g^{RR}\{\phi\phi, R\} = -R$$

$$\left\{ {\phi \atop \phi R} \right\} = \left\{ {\phi \atop R\phi} \right\} = g^{\phi\phi}\{\phi R, \phi\} = \frac{1}{R}$$

$$\Rightarrow \ddot{R} - R\dot{\phi}\dot{\phi} = 0, \quad \ddot{\phi} + \frac{2}{R}\dot{\phi}\dot{R} = 0, \quad \ddot{z} = 0$$

These are the geodesic equations. They follow from (6.68) when we sum over b and c.

7 Choosing an arbitrary coordinate system,

$$0 = \nabla_c g_{ab} = \partial_c g_{ab} - \Gamma^m{}_{ac} g_{mb} - \Gamma^m{}_{bc} g_{am}$$

$$\Rightarrow \quad \Gamma_{bac} + \Gamma_{abc} = \partial_c g_{ab}$$
$$= \Gamma_{bca} + \Gamma_{abc}$$

We have used the symmetry of ∇_c in the last line. Writing down cyclic permutations, we find

$$\begin{array}{rl} & \Gamma_{abc} + \Gamma_{bca} = \partial_c g_{ab} \\ + & \Gamma_{cab} + \Gamma_{abc} = \partial_b g_{ca} \\ - & \Gamma_{bca} + \Gamma_{cab} = \partial_a g_{bc} \\ \hline & 2\Gamma_{abc} = \partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{bc} \end{array}$$

$$\Rightarrow \quad \Gamma^a{}_{bc} = \tfrac{1}{2} g^{am} \left(\partial_b g_{cm} + \partial_c g_{bm} - \partial_m g_{bc} \right)$$

Comparing with (6.62) and (6.64) shows explicitly that ∇_c is the symmetric metric connection.

8 a) The Riemann tensor is found in (6.39) to be

$$R^a{}_{bcd} = 2\Gamma^a{}_{b[c,d]} + \Gamma^a{}_{e[c}\Gamma^e{}_{|b|d]}$$

Working in geodetic coordinates, the second term vanishes, but the first term does not. However, because $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$, we have immediately that

$$R^a{}_{[bcd]} = 2\Gamma^a{}_{[bc,d]} = 0$$

The six terms on the left reduce to the three in (6.78) because $R^a{}_{bcd} = -R^a{}_{bdc}$. To get (6.79), we write

$$R_{abcd} = g_{am}\Gamma^m{}_{b[d,c]} = \Gamma_{ab[c,d]} \qquad \text{(geodetic)}$$

$$= \tfrac{1}{2}\left(g_{ab,[c} + g_{a[c|,b|} - g_{b[c|,a|}\right)_{,d]}$$

$$= \tfrac{1}{2}\left(g_{a[c,d]b} - g_{b[c,d]a}\right) = g_{[a[c,d]b]}$$

Here, we have used the flatness of ∂_a. The final result is clearly symmetric under $ab \leftrightarrow cd$.

b) Working again in geodetic coordinates,

$$R^a{}_{bcd;e} = \Gamma^a{}_{b[c,d]e}$$

We can ignore the second term in (6.39) because, even when the extra derivative hits one of the Γ's, the other still vanishes. Now, since ∂_a is flat, we immediately find

$$R^a{}_{b[cd;e]} = \Gamma^a{}_{b[c,de]} = 0$$

(6.82) follows because $R^a{}_{bcd} = -R^a{}_{bdc}$.

9 From 6.6, we have

$$D^k{}_{ij} = \Gamma^k{}_{ij} - \Gamma'^k{}_{ij} = \nabla_{\bar{e}_i}\bar{e}_j \cdot \tilde{\omega}^k - \nabla'_{\bar{e}_i}\bar{e}_j \cdot \tilde{\omega}^k$$

Now suppose we do a basis transformation $\bar{e}_i \to \bar{e}_{i'} = \Lambda^i{}_{i'}\,\bar{e}_{i'}$, with $\Lambda^i{}_{i'}$ a collection of

6 of scalar functions. Then, using (2.34) and (2.35)

$$D^{k'}{}_{i'j'} = \nabla_{(\Lambda^i{}_{i'}\bar{e}_i)}(\Lambda^j{}_{j'}\bar{e}_j)\cdot\Lambda^{k'}{}_k\,\tilde{\omega}^k \;-\; \nabla\to\nabla'$$

$$= \Lambda^i{}_{i'}\Lambda^j{}_{j'}\Lambda^{k'}{}_k\,\nabla_{\bar{e}_i}\bar{e}_j\cdot\tilde{\omega}^k$$

$$+\, \Lambda^i{}_{i'}\Lambda^{k'}{}_k\,\nabla_{\bar{e}_i}\Lambda^j{}_{j'}\cdot\delta^k_j \;-\; \nabla\to\nabla'$$

$$= \Lambda^i{}_{i'}\Lambda^j{}_{j'}\Lambda^{k'}{}_k\,D^k{}_{ij}$$

$$+\, \Lambda^i{}_{i'}\Lambda^{k'}{}_j\,\bar{e}_i(\Lambda^j{}_{j'}) - \Lambda^i{}_{i'}\Lambda^{k'}{}_j\,\bar{e}_i(\Lambda^j{}_{j'})$$

Thus, only the first term survives because all connections act identically on scalars. We have therefore shown that $D^k{}_{ij}$ transforms as a tensor. Now take $\bar{e}_i$ to be a coordinate basis in (6.15), so that the last term on the left vanishes. Then

10 For a tensor T, define $\mathcal{D}_{\bar{U}}T$ by replacing partial derivatives with covariant ones in the Lie derivative. This operator is linear and Leibniz because ∇_a is. Moreover, for scalars f and vectors $\bar{V}$,

$$\mathcal{D}_{\bar{U}} f = \nabla_{\bar{U}} f = \bar{U}(f) = \mathcal{L}_{\bar{U}} f$$

$$\mathcal{D}_{\bar{U}}\bar{V} = \nabla_{\bar{U}}\bar{V} - \nabla_{\bar{V}}\bar{U} = \mathcal{L}_{\bar{U}}\bar{V} \qquad (\nabla \text{ symmetric})$$

Thus, $\mathcal{D}_{\bar{U}}$ and $\mathcal{L}_{\bar{U}}$ are both linear and Leibniz, and they agree on scalars and vectors. The therefore agree on all tensors.

11 a) In a coordinate basis, $[\bar{e}_i, \bar{e}_j] = 0$, so

$$R^{\ell}{}_{kij}\,\bar{e}_\ell = \nabla_{\bar{e}_i}\nabla_{\bar{e}_j}\bar{e}_k - i \leftrightarrow j$$

$$= \nabla_{\bar{e}_i}(\Gamma^m{}_{kj}\,\bar{e}_m) - i \leftrightarrow j$$

$$= \Gamma^m{}_{kj,i}\,\bar{e}_m + \Gamma^m{}_{kj}\nabla_{\bar{e}_i}\bar{e}_m - i \leftrightarrow j$$

$$= \Gamma^{\ell}{}_{kj,i}\,\bar{e}_\ell + \Gamma^m{}_{kj}\Gamma^{\ell}{}_{mi}\,\bar{e}_\ell - i \leftrightarrow j$$

This is the result.

b) For a non-coordinate basis, we must add the term

$$-\nabla_{[\bar{e}_i, \bar{e}_j]}\bar{e}_k = -\nabla_{c^m{}_{ij}\bar{e}_m}\bar{e}_k$$

$$= -c^m{}_{ij}\nabla_{\bar{e}_m}\bar{e}_k = -c^m{}_{ij}\Gamma^{\ell}{}_{km}\,\bar{e}_\ell$$

This gives the corrected result.

c) The first four terms in $R^{\ell}{}_{kij}$ are already manifestly antisymmetric in i and j. But so is $c^m{}_{ij}$, so the first result follows.

We have proved the second result in problem 8.

d) $R^{\ell}{}_{kij}$ has n^4 components subject to $\frac{n(n+1)}{2}\cdot n^2$ constraints from the first identity and $\frac{n(n-1)(n-2)}{6}\cdot n$ constraints from the second:

$$n^4 - \frac{n(n+1)}{2}n^2 - \frac{(n-1)(n-2)}{6}n^2 = n^2\left[\frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{6}\right] = n^2\frac{n^2-1}{3}$$

12 a) First, we have

$$\bar{e}_r = \frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\,\bar{e}_x + \sin\theta\,\bar{e}_y$$

$$\bar{e}_\theta = \frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} = -r\sin\theta\,\bar{e}_x + r\cos\theta\,\bar{e}_y$$

$$\Rightarrow \bar{e}_x = \cos\theta\,\bar{e}_r - \tfrac{1}{r}\sin\theta\,\bar{e}_\theta$$
$$\bar{e}_y = \sin\theta\,\bar{e}_r + \tfrac{1}{r}\cos\theta\,\bar{e}_\theta$$

Now, $\nabla\bar{e}_x = \nabla\bar{e}_y = 0$, so

$$\nabla_{\bar{e}_i}\bar{e}_r = (-\sin\theta\,\bar{e}_x + \cos\theta\,\bar{e}_y)\nabla_{\bar{e}_i}\theta = \tfrac{1}{r}\bar{e}_\theta\,\bar{e}_i(\theta)$$

$$\nabla_{\bar{e}_i}\bar{e}_\theta = (-\sin\theta\,\bar{e}_x + \cos\theta\,\bar{e}_y)\nabla_{\bar{e}_i}r$$
$$-r(\cos\theta\,\bar{e}_x + \sin\theta\,\bar{e}_y)\nabla_{\bar{e}_i}\theta$$

$$= \tfrac{1}{r}\bar{e}_\theta\,\bar{e}_i(r) - r\,\bar{e}_r\,\bar{e}_i(\theta)$$

In the first case, we must have $i=\theta$ on the right, which then gives $\Gamma^\theta{}_{r\theta} = \frac{1}{r}$. The second case allows us to read off $\Gamma^\theta{}_{\theta r} = \frac{1}{r}$ and $\Gamma^r{}_{\theta\theta} = -r$ similarly.

b) We have $\bar{V} = V^r\,\bar{e}_r + V^\theta\,\bar{e}_\theta$, so

$$\nabla_{\bar{e}_i}\bar{V} = \bar{e}_i(V^r)\,\bar{e}_r + \tfrac{1}{r}V^r\,\bar{e}_\theta\,\bar{e}_i(\theta)$$
$$+ \bar{e}_i(V^\theta)\,\bar{e}_\theta + \tfrac{1}{r}V^\theta\,\bar{e}_\theta\,\bar{e}_i(r) - rV^\theta\,\bar{e}_r\,\bar{e}_i(\theta)$$

This gives $\nabla_i V^j$. Contracting with $\tilde{\omega}^i$ gives

$$\nabla_i V^i = \bar{e}_r(V^r) + \tfrac{1}{r}V^r\,\bar{e}_\theta(\theta) + \bar{e}_\theta(V^\theta)$$
$$= \frac{\partial}{\partial r}V^r + \frac{1}{r}V^r + \frac{\partial}{\partial\theta}V^\theta$$

c) We have $\hat{e}_r = \bar{e}_r$ and $\hat{e}_\theta = \frac{1}{r}\bar{e}_\theta = -\sin\theta\,\bar{e}_x + \cos\theta\,\bar{e}_y$:

$$\nabla_{\hat{e}_i} \hat{e}_r = \frac{1}{r}\bar{e}_\theta\, \hat{e}_i(\theta) = \hat{e}_\theta\, \hat{e}_i(\theta)$$

$$\nabla_{\hat{e}_i} \hat{e}_\theta = -(\cos\theta\,\bar{e}_x + \sin\theta\,\bar{e}_y)\, \hat{e}_i(\theta) = -\hat{e}_r\, \hat{e}_i(\theta)$$

Thus, $\Gamma^{\hat{\theta}}{}_{\hat{r}\hat{\theta}} = -\Gamma^{\hat{r}}{}_{\hat{\theta}\hat{\theta}} = \frac{1}{r}$ are the only non-zero Christoffel symbols in this case.

d) Writing $\bar{V} = V^{\hat{r}}\hat{e}_r + V^{\hat{\theta}}\hat{e}_\theta$, we find

$$\nabla_{\hat{e}_i}\bar{V} = \nabla_{\hat{e}_i}V^{\hat{r}}\cdot\hat{e}_r + V^{\hat{r}}\hat{e}_\theta\,\hat{e}_i(\theta) + \nabla_{\hat{e}_i}V^{\hat{\theta}}\cdot\hat{e}_\theta - V^{\hat{\theta}}\hat{e}_r\,\hat{e}_i(\theta)$$

$$= \left(\nabla_{\hat{e}_i}V^{\hat{r}} - \frac{1}{r}V^{\hat{\theta}}\delta_i^{\hat{\theta}}\right)\hat{e}_r + \left(\nabla_{\hat{e}_i}V^{\hat{\theta}} + \frac{1}{r}V^{\hat{r}}\delta_i^{\hat{\theta}}\right)\hat{e}_\theta$$

$$\Rightarrow \nabla_i V^i = \frac{\partial}{\partial r}V^{\hat{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}V^{\hat{\theta}} + \frac{1}{r}V^{\hat{r}}$$

This is the usual divergence in polar coordinates.