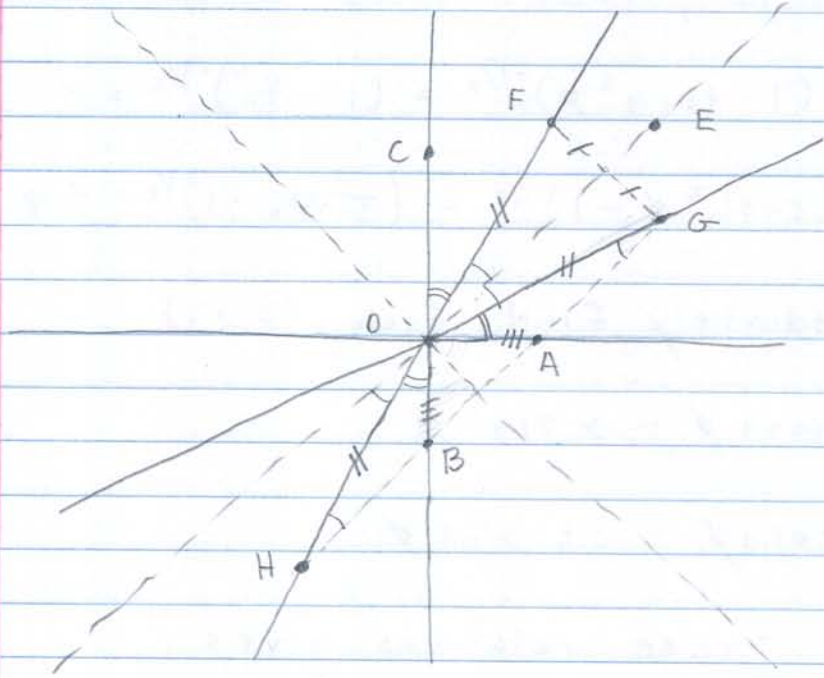
Problem set I

1



Drop a perpendicular from G to the line OE and find the point F an equal distance on the other side. OGF is isoceles. The line GH is parallel to OE, so the angle OGA is equal to EOG. AOB is a right isoceles triangle, so OA is equal to OB. Moreover, HBO and OAG are both 135° angles, and COF and HOB are also equal. Thus, HOB and GOA are identical triangles. Therefore, HO = OG = OF and the observer moving along OF will see G simultaneous with O.

Observers whose world lines intersect the segment FE will see G occur before O, the others, after.

2 If $\tanh\phi = \frac{v}{c}$, then

$$\cosh\phi = (1-\tanh^2\phi)^{-1/2} = \left(1-\frac{v^2}{c^2}\right)^{-1/2} = \gamma$$

$$\sinh\phi = (\cosh^2\phi - 1)^{1/2} = \left(\frac{1}{1-v^2/c^2} - 1\right)^{1/2} = \frac{v}{c}\gamma$$

Thus, we immediately find from (3.12)

$$ct' = ct\cosh\phi - x\sinh\phi$$

$$x' = x\cosh\phi - ct\sinh\phi$$

The sum of these relations gives

$$ct' \pm x = ct(\cosh\phi \mp \sinh\phi) - x(\sinh\phi \mp \cosh\phi)$$

$$= ct\, e^{\mp\phi} \pm x\, e^{\mp\phi} = e^{\mp\phi}(ct \pm x)$$

We also have

$$e^{2\phi} = \frac{e^{\phi}}{e^{-\phi}} = \frac{\cosh\phi + \sinh\phi}{\cosh\phi - \sinh\phi} = \frac{\gamma + \gamma v/c}{\gamma - \gamma v/c} = \frac{c+v}{c-v}$$

3 The equation of motion follows immediately when we set $x_0 = ct_0 = 0$ in (3.24) and multiply through by c^4/a.

A photon sent after the receding particle at time t_0 from the origin follows the world line $x = c(t - t_0)$. We seek simultaneous solutions of these two equations.

So, we have

$$a c^2 (t-t_0)^2 + 2c^3 (t-t_0) - a c^2 t^2 = 0$$

$$= -2ac^2 t t_0 + ac^2 t_0^2 + 2c^3(t-t_0)$$

$$= 2c^3(t-t_0) - 2ac^2 t_0 (t-t_0) - ac^2 t_0^2$$

$$\Rightarrow \quad t - t_0 = \frac{ac^2 t_0^2}{2c^2(c - a t_0)}$$

The result is physical only if $t - t_0 > 0$, which demands tha $a t_0 < c \Rightarrow t_0 < c/a$.

The second equation on p. 37 and (3.17) give

$$\frac{dv}{dt} = \left(1 - \frac{v^2}{c^2}\right)^{3/2} a \qquad \text{and} \qquad \frac{d\tau}{dt} = \left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

$$\Rightarrow \frac{dv}{d\tau} = \left(1 - \frac{v^2}{c^2}\right) a \quad \Rightarrow \quad v(\tau) = c \tanh \frac{a\tau}{c}$$

$$\Rightarrow \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \left(1 - \tanh^2 \frac{a\tau}{c}\right)^{-1/2} = \cosh \frac{a\tau}{c}$$

$$\frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \cosh \frac{a\tau}{c} \quad \Rightarrow \quad t = \frac{c}{a} \sinh \frac{a\tau}{c}$$

$$\frac{dx}{d\tau} = \left(1 - \frac{v^2}{c}\right)^{-1/2} v = c \sinh \frac{a\tau}{c} \quad \Rightarrow \quad x = \frac{c^2}{a}\left(\cosh \frac{a\tau}{c} - 1\right)$$

We have used initial conditions $\tau = 0$ at $t = 0$ and $x = 0$ to get these results.

If $t \ll \frac{c}{a}$, then we have

$$\tau = \frac{c}{a} \sinh^{-1} \frac{at}{c} = \frac{c}{a}\left[\frac{at}{c} - \frac{1}{6}\left(\frac{at}{c}\right)^3 + \cdots\right] = t\left[1 - \frac{1}{6}\left(\frac{at}{c}\right)^2 + \cdots\right]$$

4

Let $T = 1$ hour, and

$$\frac{aT}{c} = 3.5 \times 10^{-4} << 1$$

$$\Rightarrow \quad t - \tau \cong \frac{1}{6}\left(\frac{at}{c}\right)^2 t \cong 7.5 \times 10^{-5} \text{ sec}$$

When $T = 10$ days, we scale the right side by 240^3, giving $t - \tau \cong 17$ min.

4 Here, we use conservation of energy to find the 4-momentum of the composite particle:

$$E = \frac{\bar{m}_0}{\sqrt{1-v^2}} + m_0 \qquad \qquad P = \frac{\bar{m}_0 v}{\sqrt{1-v^2}} + 0$$

$$\Rightarrow M^2 = E^2 - P^2 = \bar{m}_0^{\,2} + m_0^2 + \frac{2 m_0 \bar{m}_0}{\sqrt{1-v^2}}$$

The answer in the back of the book is wrong.

5 Working in the lab frame, the four equations needed to determine the final 4-momenta of the particles arise from (a) conservation of energy and (b) the fact that the mass of each particle is unaltered:

$$E + e = M + e_0 \qquad \qquad E^2 - P^2 = M^2$$

$$P + p = p_0 \qquad \qquad e^2 - p^2 = m^2$$

We expect to find quadratic equations for P and p, whose roots correspond to the initial and final states of motion. We therefore solve for P first because one of the roots should be zero. Accordingly, we eliminate e and p using the conservation equations. The mass condition for m gives

$$m^2 = (M + e_0 - E)^2 - (P - p_0)^2$$

$$= M^2 + e_0^2 + E^2 - P^2 - p_0^2 + 2Me_0 - 2ME - 2e_0 E + 2Pp_0$$

$$= 2M^2 + m^2 + 2Me_0 - 2(M+e_0)E + 2Pp_0$$

$$\Rightarrow \quad (M+e_0)E = (M+e_0)M + Pp_0$$

We now eliminate E by squaring and using the mass condition for M:

$$(M+e_0)^2 E^2 = (M+e_0)^2 M^2 + 2(M+e_0)MPp_0 + P^2 p_0^2$$

$$\Rightarrow [(M+e_0)^2 - p_0^2]P^2 - 2(M+e_0)Mp_0 P = 0$$

$$\Rightarrow P = \frac{2(M+e_0)M}{(M+e_0)^2 - p_0^2} p_0 = \frac{2M(M+e_0)}{M^2 + 2Me_0 + m^2} p_0$$

$$\Rightarrow p = p_0 - P = \frac{m^2 - M^2}{M^2 + 2Me_0 + m^2} p_0$$

6 Here, conservation of energy gives

$$m_0 + h\nu = \gamma m \qquad \text{and} \qquad 0 + h\nu = \gamma m v$$

$$\Rightarrow \quad m^2 = (m_0 + h\nu)^2 - (h\nu)^2 = m_0(m_0 + 2h\nu), \quad v = \frac{h\nu}{m_0 + h\nu}$$