Homework II

1 L and N clearly span the plane orthogonal to that spanned by M and $\bar{M}$. Thus, we only need to check

$$L \cdot L = \tfrac{1}{2}(T \cdot T + 2T \cdot Z + Z \cdot Z) = \tfrac{1}{2}(-1 + 0 + 1) = 0$$

$$N \cdot N = \tfrac{1}{2}(-1 + 1) = 0 \qquad L \cdot N = \tfrac{1}{2}(-1 - 1) = -1$$

$$M \cdot M = \tfrac{1}{2}(X \cdot X + 2i X \cdot Y - Y \cdot Y) = \tfrac{1}{2}(1 + 0 - 1) = 0$$

$$\bar{M} \cdot \bar{M} = \tfrac{1}{2}(1 - 1) = 0 \qquad M \cdot \bar{M} = \tfrac{1}{2}(1 - i^2) = 1$$

2 Let's solve problem 7.14 from d'Inverno's book first. We have

$$\nabla_a \nabla_b X_c = \nabla_b \nabla_a X_c + R_{abc}{}^d X_d$$

$$= -\nabla_b \nabla_c X_a + R_{abc}{}^d X_d$$

We have used the definition of the curvature in the first line, and Killing's equation $\nabla_{(a} X_{c)} = 0$ in the second. The result has a cyclic permutation of indices, so we find

$$\nabla_a \nabla_b X_c = -(-\nabla_c \nabla_a X_b + R_{bca}{}^d X_d) + R_{abc}{}^d X_d$$

$$= \nabla_c \nabla_a X_b + (R_{abc}{}^d - R_{bca}{}^d) X_d$$

$$= -\nabla_a \nabla_b X_c + (R_{abc}{}^d - R_{bca}{}^d + R_{cab}{}^d) X_d$$

2 Using the first Bianch identity $R_{[abc]}{}^d = 0$ and collecting the derivatives on the left gives the final result

$$\nabla_a \nabla_b X_c = R_{cba}{}^d X_d$$

when we divide by 2 and interchange the first two indices on the curvature.

When spacetime is flat, the right side here vanishes, so

$$\nabla_a \nabla_b X_c = 0 \quad \Rightarrow \quad \nabla_b X_c = w_{bc} = \text{const.}$$

Since $\nabla_{(b} X_{c)} = 0$ by Killing's equation, w_{bc} must be anti-symmetric. Integrating again gives the result

$$X_c = x^b w_{bc} + t_c.$$

In n dimensions, w_{bc} has $\frac{1}{2}n(n-1)$ independent components, while t_c has n. Therefore, a Killing vector is specified by $\frac{1}{2}n(n+1)$ constants. In four dimensions, these ten constants describe four translations, three rotations and three boosts.

3 The unit basis vectors in the rotating frame rotate with the frame, so, for example

$$\frac{d}{dt}\hat{\imath}' = \vec{w} \times \hat{\imath}'$$

In the rotating frame, however, we take these to be constant. Thus,

$$\left[\frac{d\vec{v}}{dt}\right]_{S'} = \frac{dv_1}{dt}\,\hat{\imath}' + \frac{dv_2}{dt}\,\hat{\jmath}' + \frac{dv_3}{dt}\,\hat{k}'$$

The result follows immediately by the Leibniz rule for the products $v_1 \hat{\imath}'$, etc.

4 Let's take a different approach here. Using components in both frames, we write

$$r^\alpha \hat{e}_\alpha = s^\alpha \hat{e}_\alpha + r'^\alpha \hat{e}'_\alpha$$

Taking two derivatives, and recalling that $\dot{\hat{e}}_\alpha = 0$ and $\dot{\hat{e}}'_\alpha = \vec{\omega} \times \hat{e}'_\alpha$, we find

$$\ddot{r}^\alpha \hat{e}_\alpha = \ddot{s}^\alpha \hat{e}_\alpha + \ddot{r}'^\alpha \hat{e}'_\alpha + 2\dot{r}'^\alpha \dot{\hat{e}}'_\alpha + r'^\alpha \ddot{\hat{e}}'_\alpha$$

$$= \ddot{s}^\alpha \hat{e}_\alpha + \ddot{r}'^\alpha \hat{e}_\alpha + 2\dot{r}'^\alpha\, \vec{\omega} \times \hat{e}_\alpha$$

$$+ r'^\alpha \frac{d}{dt}\left(\vec{\omega} \times \hat{e}'_\alpha\right)$$

The result follows when we set $\vec{r}' = r'^\alpha \hat{e}'_\alpha$, $\dot{\vec{r}}' = \dot{r}'^\alpha \hat{e}'_\alpha$ and $\ddot{\vec{r}}' = \ddot{r}'^\alpha \hat{e}'_\alpha$, and recall that $\ddot{r}^\alpha = F^\alpha/m$ in the inertial frame.

5 The simplest generalization uses the torsion-free connection ∇_a:

$$\nabla_{[a} F_{bc]} = 0$$

This is because the inertial coordinate connection ∂_a in Minkowski spacetime <u>is</u> the symmetric metric connection.

Now, using the Christoffel tensor $\Gamma_{ab}{}^{c}$ for $\nabla_a - \partial_a$, where ∂_a is <u>any</u> coordinate derivative on curved spacetime, we find

$$\nabla_a F_{bc} = \partial_a F_{bc} + \Gamma_{ab}{}^{m} F_{mc} + \Gamma_{ac}{}^{m} F_{bm}$$
$$= \partial_a F_{bc} - 2\Gamma_{a[b}{}^{m} F_{c]m}$$

We have used anti-symmetry of F_{ab} in the second line. When we anti-symmetrize over a as well, however, this second term vanishes because ∇_a is torsion-free and thus $\Gamma_{ab}{}^{c} = \Gamma_{ba}{}^{c}$. Therefore,

$$\partial_{[a} F_{bc]} = \nabla_{[a} F_{bc]} = 0$$

Thus, the coordinate curl of F_{bc} vanishes in all charts in all spacetimes.